The Visible Hand: Price Discrimination under Heterogeneous Precision*

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Abstract

Technological innovations have allowed some sellers to collect detailed information about buyers. I study these changes in a standard search-theoretic model of imperfect competition, featuring buyers with heterogeneous private valuations for quality, and introduce sellers who observe valuation signals of heterogeneous precision. Signals induce third-degree price discrimination, and their precision largely dictates whether they are used to increase trade or increase markups - impacting aggregate surplus and its distribution. When buyers' valuations are more heterogeneous, imprecisely informed sellers prioritize high markups despite limiting trade, and precision relaxes this tension, not only allowing them to pursue high markups when it is least obstructive but also primarily incentivizing low markup offers that increase trade upon signals indicative of low valuation - increasing aggregate surplus and benefiting (hurting) buyers with a low (high) valuation. However, when valuations are more homogeneous, imprecisely informed sellers prioritize trade, and precision can primarily incentivize high markup offers that limit trade upon signals indicative of high valuation, hurting all buyers and even decreasing aggregate surplus. In either case, precision makes sellers more profitable, but its effect on competitors can be positive or negative. Generally, competitors suffer (benefit) when laggards (leaders) gain precision.

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1 Introduction

> **Matt Murray (Wall Street Journal, Editor in Chief):** *The perception of a lot of people is that you've morphed...In a lot of ways, you're thought of as a data company more than a retail company.*
> **Jeff Wilke (CEO, Amazon Worldwide Consumer):** *There was a corner pharmacy where I grew up. The pharmacist had been there forever. When you walked in, he knew what you liked to buy...That's the same thing we're doing. Our main purpose in storing your purchases is so that we can recommend something that you might want to buy the next time.*
>
> -"Amazon's Defense of Private Brands" (*WSJ*, 10/24/19).

Amazon's well-documented harvesting and leveraging of consumer data exemplifies a sweeping transformation in the way contemporary firms operate. A surge in the availability of data and in the power of analytical methods that uncover its insights[1] now allows firms to better discern consumer preferences and tailor offers[2]. These methods have been broadly but heterogeneously deployed across industries and functions. However, despite the importance of this pipeline, its impact on markets is not well understood.

There are several natural questions about this technological change. First, whether it benefits us in aggregate and as consumers in particular is of immediate importance. In this vein, policymakers have raised concerns about potential detrimental effects on competition (FTC (2012, 2013, 2014), CEA (2015), UK Competition and Markets Authority (2021)) and enacted wide-ranging measures, such as the EU's General Data Protection Regulation and Artificial Intelligence Act. Second, how do firms benefit from investments in prediction? The growing analytics gap[3] between Amazons of the world and more traditional businesses suggests that investment is profitable, but in the right context. So, what is that context? And, how does this type of investment affect competitors? To address these questions, I leverage a standard search-theoretic framework of imperfect competition, featuring buyers with heterogeneous private valuations for quality, and introduce sellers who observe buyer valuation signals. Signals are characterized by their precision, measured as the probability of correctly predicting a buyer's valuation, and I allow this ex-ante attribute to be heterogeneous among sellers. I characterize equilibria analytically, linking properties of information with properties of offers, and study the comparative statics of both precision and competition, documenting their effects on total surplus as well as its distribution between and within buyers and sellers.

The precision of valuation information fundamentally determines its impact on trade and the distribution of its gains. It shapes sellers' trade-off between profiting through higher sales versus higher markups, so different levels of precision give rise to different offer strategies. In particular, precision makes their offers more responsive to their signals, giving them the confidence to extend higher-markup offers that decrease sales upon signals indicative of high buyer valuations and lower-markup offers that increase sales upon signals indicative of low buyer valuations. Precision also increases the sales of any offer strategy by increasing the probability that its high- (low-) markup offers are extended to buyers with a high (low) valuation. The net result of these two effects is that precision can increase or decrease sales and, by extension, the efficiency of trade.

[1] Since 2018, the McKinsey Global Institute has conducted a yearly survey on the "State of AI", "representing the full range of regions, industries, company sizes, functional specialties, and tenures". Approximately half of all firms consistently report the adoption of AI in at least one business function, while the average number of functions has doubled since the first survey.

[2] Mikians et al. (2012, 2013), Hannak et al. (2014), Chen et al. (2015) document the pervasiveness of these methods among sellers, both large and small, while industry surveys by Deloitte (2018) and McKinsey (2023) echo these points, respectively finding widespread use of AI for personalization and that "Marketing and Sales" along with "Product/Service Development" are the most common business functions for AI applications.

[3] In McKinsey's surveys, high-performing organizations are more than three times as likely to report that their data and analytics contributed at least 20% of earnings before interest and taxes.

When buyers' valuations are more heterogeneous, imprecisely informed sellers pursue inefficient high-markup strategies, and additional precision primarily relaxes this motive; however, when buyers' valuations are more homogeneous, imprecisely informed sellers pursue efficient high-sale strategies, and additional precision can weaken this motive. Beyond its aggregate impact, precision is inherently redistributive, shifting the share of surplus between buyers and sellers as well as its distribution within each side of the market. On the demand side, all buyers benefit (suffer) when precision strongly stimulates (contracts) trade, whereas if it only moderately stimulates trade then high-valuation buyers - the principal targets of high markups - suffer and low-valuation buyers - the principal victims of rationed trade - benefit. On the supply side, precision improves the profitability of sellers, but it can make competitors more or less profitable. This is because precision softens competition for high-valuation buyers but intensifies competition for low-valuation buyers, and either effect can dominate. I find that economies with a high level of competition and valuation heterogeneity are most conducive to negative profit externalities from improvements in peer precision. Precision heterogeneity introduces a particularly interesting twist to these externalities, as it reveals that advances in the precision of laggards are generally detrimental for leaders, whereas all sellers benefit when leaders surge further ahead.

I confirm the traditional effects of competition but also find a new - concerning - one. Characteristically, competition increases trade (efficiency) and buyer surplus. However, competition and imprecision both reduce the sensitivity of sellers' offers to their signals. This complementarity makes less precisely informed sellers particularly prone to forego using their predictive skill and implies that competition can exacerbate the documented disparity in the use of predictive technologies.

Before proceeding with more detailed results, it is helpful to describe the components of the model. On one side of the market, there is a unit mass of buyers with low or high private valuations for the quality of a good. On the other side, there is a unit mass of sellers who produce the good with a common technology. I introduce imperfect competition by matching buyers and sellers in the style of Burdett and Judd (1983): each buyer matches with either one or two sellers who make simultaneous take-it-or-leave-it offers, without any information about the number of competitors in the match beyond the commonly known matching protocol. In this formulation, I capture the full range of competitive intensity, from monopoly to perfect competition, by varying the probability of matching with only one seller. My innovation is (a) introducing additional seller information about buyers' valuations with flexible precision and (b) allowing precision to differ among sellers. Concretely, I model this by assuming that the population distribution of buyer valuations is common knowledge but that each seller observes a private signal about the matched buyer's valuation. Signals can take two possible values, low or high, and their precision is the probability of taking a low (high) value when the matched buyer has a low- (high-) valuation. A discretization with two levels of precision suffices to study seller heterogeneity: that of less precisely informed sellers, referred to as *amateurs*, and that of more precisely informed sellers, referred to as *sharks*. This allows me to capture the full range of valuation information asymmetry between buyers and sellers as well as between sellers, by varying the absolute levels of amateur and shark precision[4].

I introduce most of the core ideas in a simpler environment, where sellers have identical precision and offer an identical good of exogenously specified quality, so predictive skill only orients pricing. Then, I proceed to the general environment where sellers have differing precision and can choose the quality of the goods that they offer, so predictive skill orients pricing and production. In all economies, equilibria are structured by several ordering relations. A principal one relates to the conditional

[4]I could also study the effects of precision by varying the proportion of amateur and shark sellers, but the analysis would be analogous and less direct.

profitability of offers. In particular, high-markup offers are more profitable in matches with high-valuation buyers, whereas high-sale offers, which share more of the gains from trade with buyers, are more profitable in matches with low-valuation buyers. This equilibrium relation is generated by the privacy and heterogeneity of preferences, so it would exist even without valuation signals, but it helps align the differing priorities that signals induce among sellers. To understand this point, recall the timing of this incomplete information game: sellers match with buyers, observe a valuation-relevant signal, and update their beliefs. At this stage, a seller's type is summarized by its posterior probability that the buyer has a low-valuation. The conditional profitability relation therefore incentivizes sellers of a larger type to extend higher sale offers that are more attractive to all buyers and also more efficient. Since a seller's type is greater when it observes a low (versus high) signal and when its signals are more precise, sellers with lower signals extend more attractive offers, while sellers with more precise information extend more extreme ones; in other words, sharks who observe high signals make the highest markup/least attractive/most inefficient offers, sharks who observe low signals make the lowest markup/most attractive/most efficient offers, and amateurs extend offers that are intermediate in these respects. In this sense, the precisions and realizations of signals order sellers' types (by Bayes' rule), and sellers' types order offers' implied surplus/efficiency (by equilibrium incentives). The ultimate effect of a change in precision on trade and welfare is the net of its effect on the distribution of matches between each type of buyer and seller, and its effect on the offers of each type of seller extends. These offer and matching effects can go in opposite directions and generate non-monotone comparative statics for precision. This is why it is imperative to consider information counterfactuals beyond the traditional extremes of uninformative versus perfectly informative signals. The model provides a tractable framework for this more nuanced analysis, allowing us to identify precision from pricing and production decisions, and delivering clear predictions about the aggregate and distributional effects of advances in prediction under a broad set of demand and competition environments.

Surprisingly, despite the relevance of predictive skill for the types of products that buyers and sellers trade, its effect on the efficiency of trade and consumer surplus is analogous when sellers endogenously choose quality and when they do not. This is because when predictive skill only orients pricing (exogenous quality economies), the link between markups, sales, and consumer surplus is still present, generating analogous offer and matching effects for precision. However, when the production channel is present (endogenous quality economies), equilibria change in form as sellers combine signal valuation information with that gleaned by screening. They offer menus and their mix of offers depends on their signal realization; in other words, sellers perform second- and third-degree price discrimination concurrently. The principal difference between the relationship of precision with trade in each of these two types of economies is that when quality is endogenous, trade also varies at the intensive margin (the amount of quality traded in each match). Ultimately, endogenizing quality not only allows me to assess the robustness of comparative statics but also to obtain the additional structure that makes it tractable to study precision heterogeneity. I can then appropriately characterize the impact of advances in prediction on sellers' profits by considering counterfactuals that involve changes in the precision of only a subset of sellers, isolating its spillovers on competitors.

Literature Review - My approach extends a tradition that leverages the canonical search-theoretic framework of Burdett and Judd (1983), which teaches us that sellers' incomplete information about competition can induce price dispersion and characterizes the impact of competition's expected level on the distribution of prices. Garrett, Gomes, and Maestri[5] (2019) introduce buyers with heteroge-

[5]Contemporaneously, Lester et al. (2019) study the flipside of Garret et al.'s setting - a lemons problem with an

neous valuations for a vertically differentiated good to this setting, so sellers screen and competition's expected level then also impinges on the efficiency of trade. I advance this literature by studying the effects of the level of asymmetric valuation information between buyers and sellers (through the precision of signals) as well as between different sellers (through the heterogeneity of precision).

Generally, I contribute to the extensive literature on price discrimination, particularly the strand concerned with understanding the factors that dictate the welfare implications of second- and third-degree price discrimination. The latter's tradition extends back to Pigou (1920), whereas the former has been an active field since the seminal work of Mussa and Rosen (1978). A salient theme has been the double-edged nature of price discrimination: potentially increasing efficiency but also redistributing surplus. My concern for the levers that determine the sign of these effects is in line with research that has analyzed the influence of market structure[6] and demand characteristics[7]. I contribute to this tradition in two ways. First, I characterize the dependence of price discrimination's effects on novel structural factors: mainly, preference heterogeneity, search behavior, and information precision. The role of precision is particularly important because the statistical properties of information dictate the market segments that it generates and so the characteristics of demand in each segment, which this literature highlights as central to the welfare effects of third-degree price discrimination. My second contribution is linking second- and third-degree price discrimination, as sellers practice both concurrently when quality is endogenous. This methodological step has immediate practical dividends, yielding a rich set of relations between observables and (unobservable) precision, highlighting the qualitatively different profit externalities from precision advances of laggards versus leaders, and drawing attention to the point that competition can exacerbate disparities in usage of prediction technologies.

My work also forms part of a growing literature that investigates the microfoundations and macroeconomic implications of the prediction pipeline. At the data collection stage, research[8] has focused on the incentives of buyers to disclose information about their preferences, trading off the attractiveness of product offers with the attractiveness of their prices or privacy costs, while at the insight extraction stage, the role of intermediaries and their use of information design to pursue various objectives, including aggregate and distributional goals[9], has been a principal concern. These works are complementary to ours since they investigate the process of generating information and its welfare effects in a specific market setting, meanwhile, I study the welfare effects of different types of information - parameterized by an interpretable statistical property (precision) - across market settings of varying demand and supply characteristics. Relatedly, the connection between data and market structure has been studied through a multidisciplinary effort that incorporates frameworks from macroeconomics, industrial organization, and finance[10]. I share some of the qualitative welfare effects of information and competition, but microfounding the heterogeneity on both sides of the market allows us to parse their composition (finding that different types of buyers can experience opposite outcomes) and consider a larger class of information counterfactuals (finding significant differences from advances in leader and laggard precision), while my concern for the use of information to price discriminate introduces fundamentally different determinants of welfare effects.

The paper proceeds as follows: in Section 2, I introduce the environment, in Section 3, I study economies where sellers have homogeneous precision and quality is exogenous, and in Section 4, I

analogous matching mechanism where privately informed sellers obtain bids from uninformed buyers.

[6]See Holmes (1989), Armstrong and Vickers (2001), Rochet and Stole (1997, 2002), Stole (2007), Vives (2011), and Rhodes and Zhou (2022).

[7]See Robinson (1933), Schmalensee (1981), Varian (1985), and Aguirre et al. (2010)

[8]See Braghieri (2019), Ichihashi (2020), Bonatti and Cisternas (2020), Hidir and Vellodi (2021), and Ali et al. (2023).

[9]See Bergemann et al. (2015, 2018), Elliot et al. (2021), Guo et al. (2022), Yang (2022), Haghpanah and Siegel (2023), Galperti et al. (2023), and Ichihashi and Smolin (2023).

[10]See Begenau at al. (2018), Agrawal et al. (2018, 2019), Kehoe et al. (2020), Farboodi and Veldkamp (2022), and Eeckhout and Veldkamp (2022)

study economies where sellers have heterogeneous precision and quality is endogenous. The Appendix is reserved for primarily technical discussion.

2 Environment

There is a unit mass of buyers with single-unit demands and heterogeneous tastes for the quality of a good.

Assumption 2.1 (Additively Separable Utility)**.** *A buyer with a marginal value for quality θ_i obtains utility $u(q, x; \theta_i) = \theta_i q - x$ from consuming a good of quality q at a price of x.*

The utility of not trading is normalized to zero.

Sellers can produce a good of quality q using a common cost function for quality $\phi(q)$. Upon a successful transaction with a buyer for a contract (q, x), sellers receive profits $x - \phi(q)$. Buyers and sellers match according to an exogenously defined process in the style of Burdett and Judd (1983). In particular, a buyer matches with one or two randomly and independently drawn sellers, and the probability of a single match is $\tilde{\rho} \in (0, 1)$. Because I assume that the number of matches is private to the buyer, sellers face uncertainty as to whether they are in competition with another seller in any match. As such, matched sellers assign a probability of,

$$\rho = \mathbb{P}(\text{single match}|\text{matched}) = \frac{\tilde{\rho}}{\tilde{\rho} + 2(1 - \tilde{\rho})} \tag{2.1}$$

to not competing, and ρ effectively governs the level of competition - ranging from the extremes of perfect competition ($\rho = 0$) to monopoly ($\rho = 1$).

While the ex-ante distribution of buyer valuations is common knowledge, sellers also observe a private signal about the valuation of each buyer with whom they match, and the precision of signals is allowed to be heterogeneous among sellers[11]. I study the simplest discretization of precision, where sellers are either less precisely informed *amateurs* or more precisely informed *sharks* with respective precisions

$$\alpha_e = P^e(\text{signal} = i|\text{matched buyer's valuation} = \theta_i) \qquad i \in \{l, h\}, e \in \{a, s\} \tag{2.2}$$

$$0.5 \leq \alpha_a < \alpha_s < 1 \tag{2.3}$$

The two relative precision categories of amateur and shark remain fixed throughout the paper, even if the absolute level of precision that they represent differs, and the share of α_e precision sellers is denoted by $\mu(e)$.

Agents do not make choices before matching, so this setting is that of an incomplete information game that is solved at the interim stage, after matches have formed and sellers have observed their signals. A seller's type is then its posterior belief of being in a low-valuation match, $p_L^{e,j} = P^e(\theta = \theta_L|j)$,

$$p_L^{e,j} = \frac{P^e(j|\theta_L)\mathbb{P}(\theta_L)}{P^e(j|\theta_L)\mathbb{P}(\theta_L) + P^e(j|\theta_H)\mathbb{P}(\theta_H)} \tag{2.4}$$

I assume signals are essentially pairwise independent conditionally on the buyer's valuation, so they are only informative about the type of possible competitors in a match through their information about the buyer's valuation.

[11] In practice, precision can be improved by acquiring more/better data or by improving the methods with which data are analyzed. Precision heterogeneity is, therefore, natural when we consider the diversity of firms' data resources and analytical practices.

Having introduced the environment, I will first analyze the role of precision in environments where sellers have identical precision and produce a good of exogenously determined quality, so equilibrium offers take the simple form of a quoted price per unit x. This will allow me to swiftly identify some of the principal effects of competition and precision on prices and welfare. Subsequently, I will introduce heterogeneity in precision and endogenize quality, allowing sellers to choose the quality of the goods they offer. This will allow me to analyze the connection between the level and distribution of precision with the fit of products available to buyers as well as its profit externalities on competitors, but requires considering a more complex set of offers, as these, optimally, take the form of screening menus composed of quality-price contracts $((q_L, x_L), (q_H, x_H))$.

3 Exogenous Quality Setting

For the remainder of this section, I assume that sellers produce a homogeneous good of exogenous quality at zero cost.

Assumption 3.1 (Exogenous Quality Assumptions)**.** *The traded good has quality $q = 1$ and a cost of production $\phi(1) = 0$.*

As such, there are gains from trade with both types of buyers, and buyers simply choose the cheapest offer that is below their valuation, when one is available. The exclusion of product design (quality) choices allows me to isolate the role of the pricing channel, but still characterize some of the main relationships between precision, the properties of offers, and welfare that extend to settings where quality is endogenous. To obtain these insights in the most straightforward fashion, I will also assume that sellers have identical precision.

Assumption 3.2 (Homogeneous Sellers)**.** *All sellers observe signals of precision α.*

There are only two types of sellers then: sellers who observe a high signal, of type p_L^h, and sellers who observe a low signal, of type p_L^l.

In equilibrium, sellers who are more convinced that they are matched with high-valuation buyers - those who observe high signals - will offer higher prices, often beyond the willingness to pay of low-valuation buyers, and thus forego trade in some matches. Inversely, sellers who are more convinced that they are matched with low-valuation buyers - those who observe low signals - will offer lower prices, often substantially below the willingness to pay of high-valuation buyers. Both types of mispricing decrease profits, but pricing out low-valuation buyers also decreases trade and, thus, efficiency.

Greater precision will make any *given* pricing strategy more profitable and efficient by reducing instances of mispricing; however, it will also induce sellers to *change* their pricing strategies. Precision gives sellers the confidence to offer larger and more frequent discounts upon a signal indicative of low-valuation, but also to offer larger and more frequent price hikes upon a signal indicative of high-valuation. While the former increases trade, the latter decreases it. Naturally, any adjustment in strategy is towards greater profitability, and, in equilibrium, precision is individually beneficial for sellers, but its externalities on other agents (buyers or competitors) can be positive or negative.

We will learn that the effect of greater precision on trade is closely linked to the heterogeneity of buyers' valuations $\theta_H - \theta_L$, i.e. dispersion in willingness to pay. When buyers' valuations are sufficiently heterogeneous, trade will increase alongside precision, whereas if they are sufficiently homogeneous, they will go in the opposite direction. Accounting for the winning and losing sides, all/low-valuation/no buyers will obtain greater surplus when precision strongly-increases/weakly-increases/decreases trade, respectively, whereas sellers will become more profitable due to collective precision advances so long as it is not the case that both competition and valuation dispersion is high. Before characterizing the determinants of these welfare effects, I will formally introduce the seller's problem and the structure of equilibria.

3.1 Seller Problem and Equilibrium Concept

The expected profits of a type p_L^j seller from a price offer x are the product of its profits per sale, x, times its expected sales, $\mathbb{P}(\text{sale at price } x|j)$,

$$\Pi(x) = \mathbb{P}(\text{sale at price } x|j)x \tag{3.1}$$

As in private value auctions, sellers know the value of winning at the time an offer is extended, the profits per sale x, but must infer the probability of winning. I call this probability the offer's expected *sales*. It reflects the uncertainty a seller has about four variables: (1) the buyer's valuation, (2) whether the buyer is matched with another seller, (3) the type of that seller, and (4) the offer it would make. The optimal offers of a type p_L^j seller are thus the prices,

$$x^j(\mathbb{P}(\cdot|j)) = \underset{x\in\mathbb{R}}{\operatorname{argmax}}\, \Pi(x)\mathbb{P}(\text{sale at price } x|j)x \tag{3.2}$$

and its strategy is a cumulative distribution function $F^j(x)$ over these. This type of seller's expected sales are decomposed as,

$$\begin{aligned}
\mathbb{P}(\text{sale at price } x|j) &= \rho\mathbb{P}(\text{sale at price } x|\text{monopoly match}, j) \\
&\quad + (1-\rho)\Bigg(\mathbb{P}(\theta_L|\text{competitive match}, j)\mathbb{P}(\text{sale at price } x|\theta_L, \text{competitive match}, j) \\
&\quad + \mathbb{P}(\theta_H|\text{competitive match}, j)\mathbb{P}(\text{sale at price } x|\theta_H, \text{competitive match}, j)\Bigg) \\
&= \rho(p_L^j \mathbb{1}(x \le \theta_L) + p_H^j) + (1-\rho)\Bigg(p_L^j(1 - F(x|\theta_L))\mathbb{1}(x \le \theta_L) + p_H^j(1 - F(x|\theta_H))\Bigg)
\end{aligned} \tag{3.3}$$

where $F(x|\theta_i)$ is the distribution of prices the seller expects a competitor to offer in matches with θ_i valuation buyers. This decomposition quantifies the seller's sources of uncertainty: the buyer's valuation (p_i^j terms), the existence of competing offers (ρ terms), and their exact terms ($F(x|\theta)$ terms). The distribution of competing offers, $F(x|\theta)$, encapsulates uncertainty about both the type of competitors and the offers of each type, emerging naturally as an average of each type of seller's strategy,

$$\begin{aligned}
F(x|\theta) &= \sum_{j\in\{l,h\}} \mathbb{P}(\text{seller observes } j|\theta)\mathbb{P}(\text{seller offers a price below } x|\text{signal } j) \\
&= \sum_{j\in\{l,h\}} \mathbb{P}(j|\theta)F^j(x)
\end{aligned} \tag{3.4}$$

This is sufficient to define the equilibrium concept.

Definition 3.1 (Bayes-Nash Equilibrium)**.** *A Bayes-Nash equilibrium is a pair of distributions* (F^l, F^h), *for each type of seller* p_L^j, *satisfying* $\sup \mathbb{P}(F^j) \subseteq x^j(\mathbb{P}(\cdot|j))$ *for* $j \in \{l, h\}$.

Equilibria have several properties that make them tractable and intuitive. Equilibrium distributions are continuous, almost everywhere differentiable, and strictly increasing on at most two convex intervals. Furthermore, equilibria are essentially unique in the sense that profitability, buyer surplus, and efficiency are invariant across them. Their qualitative properties highlight the role of information, with the organizing principle that higher prices are more profitable in high-valuation matches and are therefore offered by sellers whose posteriors place more weight on being matched with a high-valuation buyer.

3.2 Equilibrium Structure

3.2.1 Corner Cases: Monopoly and Perfect Competition

It is instructive to begin the analysis by focusing on the two extremes of competition: monopoly ($\rho = 1$) and perfect competition ($\rho = 0$). These easily yield fundamental insights that extend to economies where competition is imperfect ($\rho \in (0,1)$) while also motivating their study.

Monopoly - A monopolist's pricing choice involves the traditional trade-off between sales and markups. A matched buyer either accepts the monopolist's offer to buy the good for a price of x and obtains $\theta_i - x$ utility, or rejects it and obtains zero utility. The monopolist's optimal take-it-or-leave-it offer therefore makes the lowest valuation buyer with whom it trades indifferent between accepting or rejecting: it sets a sales maximizing price of $x = \theta_L$, and trades with every buyer, or a markup maximizing price of $x = \theta_H$, and only trades with high-valuation buyers, choosing the offer with the highest expected revenue (costs are zero),

$$\theta_L \lesseqgtr (1 - p_L^j)\theta_H \iff p_L^j \theta_L \lesseqgtr (1 - p_L^j)(\theta_H - \theta_L)$$

The monopolist's valuation beliefs, summarized by p_L^j, determine her price offer upon signal j. If she is sufficiently convinced that the buyer's valuation is low (high p_L^j), the low price $x = \theta_L$ is optimal, since the expected value of its additional sales, $p_L^j \theta_L$, is greater than the expected cost of its discount, $(1 - p_L^j)(\theta_H - \theta_L)$; otherwise, if she is sufficiently convinced that the buyer's valuation is high, she expects extraction of their information rents to be the most profitable action and chooses a price of $x = \theta_H$.

Proposition 3.1 (Monopoly). *If $\rho = 1$, a seller of type p_L^j offers a price*[12]*,*

$$x^*(p_L^j) = \begin{cases} \theta_H & \text{if } p_L^j < p_L^* \\ \theta_L & \text{otherwise} \end{cases} \tag{3.5}$$

for the threshold probability,

$$p_L^* = \frac{\theta_H - \theta_L}{\theta_H} \tag{3.6}$$

The first general takeaway is that price dispersion increases with precision, as the posteriors of sellers with more precise information (larger α) are more dispersed ($p_L^l - p_L^h$ increases), and posterior dispersion increases the likelihood that on a low versus high signal the seller is above versus below the threshold p_L^*, respectively. In other words, precision dictates whether sellers practice third-degree price discrimination or make signal-invariant offers. A monopolist who does not price discriminate sets a uniformly low or high price, so a precision-induced switch to price discrimination impacts trade efficiency and buyer welfare very differently, depending on which strategy would have been chosen when it is less precisely informed.

When buyers' valuations are sufficiently heterogeneous, imprecisely informed sellers face severe adverse selection, so they set a uniformly high price of θ_H and only trade with high-valuation buyers.

[12] I break profitability ties in favor of trade, but that is immaterial.

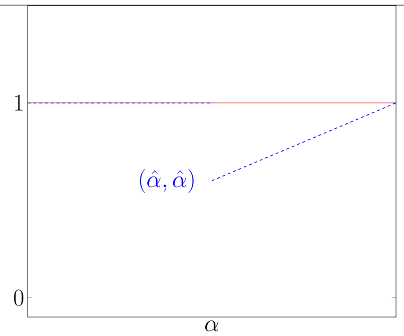
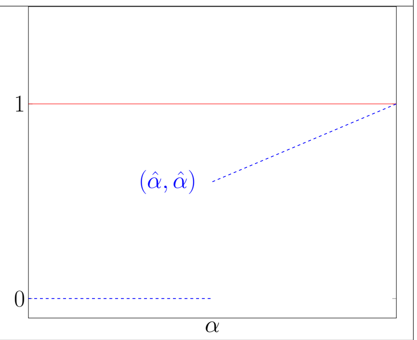
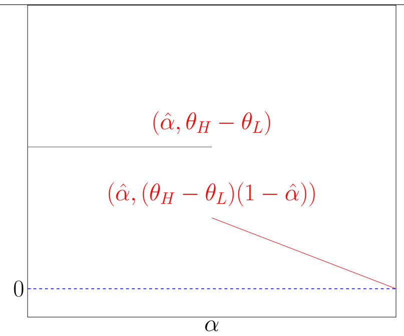
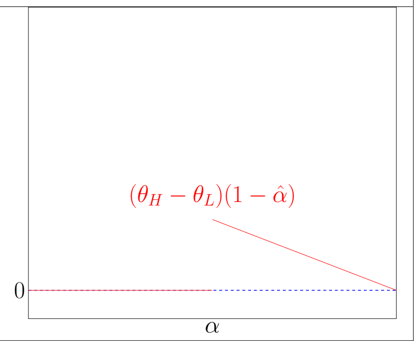
Valuation Heterogeneity $\theta_H - \theta_L$ / Y-axis Variable	**Low**	**High**
Probability of Trade: $\mathbb{P}(x \leq \theta_i \mid \theta_i)$		
Buyer Surplus: $\mathbb{E}[\, u_i \,]$		

Table 1: Consider a setting buyers always match with a single seller ($\tilde{\rho} = \rho = 1$). Low (dashed) versus High valuation buyer outcomes. At precision $\hat{\alpha}$, the seller switches from uniform pricing to third-degree price discrimination.

They only have an incentive to offer a low price of θ_L upon a low signal, when signals are precise enough, so precision is crucial to facilitating trade in these settings. A switch to price discrimination not only increases trade with low-valuation buyers - the gains of which are entirely captured by sellers - but also allows high-valuation buyers to obtain some information rents from instances of mispricing (with probability $1 - \alpha$ in each match). In this sense, better prediction can weakly benefit all agents when buyers are sufficiently heterogeneous.

However, when buyers' valuations are relatively homogeneous, imprecisely informed sellers face little adverse selection, so they set a uniformly low price of θ_L and trade with both types of buyers. Instead, they only have an incentive to offer a high price of θ_H upon a high signal, when signals are precise enough, but the switch to price discrimination also introduces mispricing. Low price offers to high-valuation buyers simply redistribute - transfering seller surplus to high-valuation buyers - but high price offers to low-valuation buyers are inefficient and decrease aggregate surplus. In this sense, advances in prediction can be inefficient and exclusively benefit sellers when buyers are sufficiently homogeneous. These effects of precision on offers and the link between its welfare effects with valuation heterogeneity generalize to settings with some (but still imperfect) competition.

Unfortunately, this simple setting of monopoly constrains the analysis in two important respects. When it comes to buyers, low-valuation buyer surplus is trivial (zero); whereas, under some competition, these buyers are able to obtain some of the gains from trade. When it comes to sellers, monopoly shuts down the profit externalities of advances in prediction.

Perfect Competition - At the other corner is perfect competition ($\rho = 0$), where sellers engage in Bertrand price competition and the unique equilibrium is one in which they offer the good at cost and buyers capture all gains from trade.

Proposition 3.2 (Perfect Competition)**.** *If $\rho = 0$, the unique Bayes-Nash equilibrium is for almost every seller to price the good at cost, $x = 0$, with probability 1.*

Therefore, perfect competition constrains the analysis of precision in an even more severe respect, mainly by making it irrelevant. However, this setting does highlight the classic pro-efficiency and pro-consumer surplus effects of competition that generalize to others where sellers have market power.

3.2.2 Interior Cases: Imperfect Competition

I will now introduce the structure of equilibria in the remaining competitive settings[13]. It is helpful to begin by focusing on the distributions of prices offered in matches with each type of buyer $\{F(x|\theta)\}_{\theta\in\{\theta_L,\theta_H\}}$. These distributions are averages of each type's strategy, so their supports are identical. They start at the top with the highest equilibrium price. This price only allows sellers to trade when they are in a monopoly match, so it is offered by sellers who expect the greatest sales from high prices (sellers who observe high signals) and is equal to their monopoly offer. Analogously, if any seller offers prices that low-valuation buyers find acceptable (individually rational), then the highest one is θ_L, as it only beats unacceptable ($x > \theta_L$) offers,

Proposition 3.3 (Highest Prices)**.** *The highest overall equilibrium price,*

$$\overline{x} = \begin{cases} \theta_H & \text{if } p_L^h \leq p_L^* \\ \theta_L & \text{otherwise} \end{cases} \tag{3.7}$$

for p_L^* from (3.6)*. And, if any prices $\leq \theta_L$ are offered, the greatest one is θ_L.*

It turns out that when high signal sellers would always offer prices that allow trade, there exists a type-invariant equilibrium where both types of sellers have the same strategy and, due to the fact that expected sales are then identical in matches with a buyer of either valuation, valuation signals do not play a role. I select this equilibrium in such settings and derive it in the Appendix, but proceed to analyze economies where precision is certain to have equilibrium effects at the margin.

Assumption 3.3 (Some Separating Offers)**.** *Precision α is such that $p_L^h \leq p_L^*$.*

In these economies, prices are supported on, at most, two convex intervals: one of prices that are only acceptable to high-valuation buyers, $[\hat{x}, \theta_H]$ with $\theta_L < \hat{x}$, and another of prices that are acceptable to all buyers.

Proposition 3.4 (Continuous Distributions, Support Convexity)**.** *Given Assumption 3.3, distributions of offers $\{F(x|\theta)\}_{\theta\in\{\theta_L,\theta_H\}}$ are atomless and supported on at most two convex intervals.*

$$\text{supp}(F(x|\theta)) = [\hat{x}, \theta_H] \quad \textit{or} \quad [\underline{x}, \theta_L] \cup [\hat{x}, \theta_H] \tag{3.8}$$

where $\theta_L < \hat{x}$.

These properties are necessary to rule out the usual deviations: atomlessness rules out discrete sales increases in competitive matches from infinitesimal discounts, convexity rules out price hikes that increase profits-per-sale without sacrificing sales, and $\theta_L < \hat{x}$ is then necessary because expected sales increase discretely as $x \searrow \theta_L$ from trade with low-valuation buyers in monopoly matches.

[13] Commentary that fills in details and some proofs are in the Appendix

It is conceptually important to highlight that offers above low-valuation buyers' valuation separate buyers - a buyer who accepts it must be of high-valuation - whereas offers below pool buyers - a buyer who accepts it can have either valuation. Optimal separating offers are thus offers that are optimal in high-valuation matches.

Proposition 3.5 (Offer Distribution over Separating Prices)**.** *The distribution of offers over separating prices is uniquely determined by the equation,*

$$\rho\theta_H = \Pi(\theta_H|\theta_H) = \Pi(x|\theta_H) = (\rho + (1-\rho)(1 - F(x|\theta_H)))x \quad \text{for } x > \theta_L$$

Stronger still, separating offers make zero profits in low-valuation matches, so equilibrium separating offers must be strictly more profitable than any pooling offer in high-valuation matches (otherwise, they would be strictly dominated) and inversely for equilibrium pooling offers, which must be strictly more profitable in low-valuation matches than any separating offer.

Proposition 3.6 (Conditional Profitability)**.** *Given an equilibrium offer $x \in \text{supp}(F(x|\theta))$, if $\theta_L < x$,*

$$\Pi(\tilde{x}|\theta_H) < \Pi(x|\theta_H) = \Pi(x'|\theta_H) \quad \forall\, \tilde{x} \leq \theta_L, x' \in \text{supp}(F(x|\theta)) \cap [\theta_L, \theta_H] \tag{3.9}$$

whereas if $x \leq \theta_L$,

$$\Pi(\tilde{x}|\theta_L) < \Pi(x|\theta_L) \quad \forall\, \theta_L < \tilde{x} \tag{3.10}$$

We see then that both types of sellers agree on the optimal separating offers and regard these as the most profitable in high-valuation matches, but sellers who observe low signals care more about profitability in low-valuation matches, so they offer lower pooling prices when the profitability edge of these in such matches and their concern for them is sufficient.

Naturally, any trade-off between profitability in low- versus high-valuation matches that makes one type of seller indifferent is strictly preferred or disfavored by the other type, so, in an equilibrium with separating offers, every offer separates, only high signal sellers' offers separate, or only low signal sellers' offers pool. The case that holds is determined by comparing the profitability that each type of seller p_L^j would expect from a separating offer, all of which are equally profitable, versus the highest pooling offer of θ_L, which would beat all separating offers.

Proposition 3.7 (Separating versus Pooling Sellers)**.** *Sellers' offers are classified by the following conditional profitability inequalities,*

if $\theta_L < p_H^l \rho\theta_H$ ***then***

All high signal seller offers separate. All low signal seller offers separate.

else if $(\rho + (1-\rho)(p_H^l\alpha + p_L^l(1-\alpha)))\theta_L < p_H^l\rho\theta_H \leq \theta_L$ ***then***

All high signal seller offers separate. Some low signal seller offers separate and some pool. The probability that a low signal seller pools, $F^l(\theta_L)$, is given by,

$$\Big(\rho + (1-\rho)(p_H^l(\alpha + (1-\alpha)(1 - F^l(\theta_L))) + p_L^l((1-\alpha) + \alpha(1 - F^l(\theta_L))))\Big)\theta_L = p_H^l\rho\theta_H \tag{3.11}$$

else if $(\rho + (1-\rho)(p_H^h\alpha + p_L^h(1-\alpha)))\theta_L < p_H^h\rho\theta_H$ ***then***

All high signal seller offers separate. All low signal seller offers pool.

else if $\rho\theta_L < p_H^h\rho\theta_H \leq (\rho + (1-\rho)(p_H^h\alpha + p_L^h(1-\alpha)))\theta_L$ ***then***

Some high signal seller offers separate and some pool. All low signal seller offers pool. The probability that a high signal seller pools, $F^h(\theta_L)$, is given by,

$$\Big(\rho + (1-\rho)(p_H^h\alpha + p_L^h(1-\alpha))(1 - F^h(\theta_L))\Big)\theta_L = p_H^h\rho\theta_H \tag{3.12}$$

else

all high signal seller offers pool / all low signal sellers pool

end if

This coarse grouping of prices is the result of a familiar ordering principle - sellers who place more posterior weight on low-valuation matches extend offers that are more attractive (on average) to all buyers - that is characteristic of economies in which sellers have predictive skill - heterogeneous or not.

Moving onto the offer distributions over pooling prices, there are two cases to consider: economies where only low signal sellers extend pooling offers and economies where both types of sellers do. In the former, low and high signal sellers' strategies have disjoint supports, so offer distributions $F(x|\theta)$ keep low signal sellers indifferent by making lower pooling prices more (less) profitable in low- (high-) valuation matches.

Proposition 3.8 (Only Low Signal Sellers Pool)**.** *If only low signal sellers extend pooling offers, offer distributions are uniquely determined by the profitability,*

$$\begin{aligned}\Pi(\theta_L; p_L^l) &= \left[p_L^l(\rho + (1-\rho)(1 - F(\hat{x}|\theta_H))) + p_H^l(\rho + (1-\rho)(1 - F(\hat{x}|\theta_L)))\right]\theta_L \\ = \Pi(x; p_L^l) &= \left[p_L^l(\rho + (1-\rho)(1 - F(x|\theta_L))) + p_H^l(\rho + (1-\rho)(1 - F(x|\theta_H)))\right]x\end{aligned}$$

and aggregation equations, $F(x|\theta_L) = \alpha F^l(x), F(x|\theta_H) = (1-\alpha)F^l(x)$*. Furthermore, given two prices* $x < x' \leq \theta_L$*,*

$$\Pi(x|\theta_H) < \Pi(x'|\theta_H) \quad , \quad \Pi(x|\theta_L) > \Pi(x'|\theta_L)$$

This conditional profitability relation is necessary because there are more low signal sellers in low-valuation matches, so discounts forfeit the same amount of revenue in monopoly matches, but provide a larger sales increase in low-valuation matches. If high signal sellers offer pooling prices, however, the support of strategies necessarily[14] overlaps over every pooling price offered by a high signal seller, $[\underline{x}^h, \theta_L]$, and offer distributions keep both types of sellers indifferent by maintaining the conditional profitability, $\Pi(x|\theta)$, of these prices. This overlapping region is followed below by a contiguous interval of prices that are only offered by low signal sellers, and over which conditional profitability trends are as per the logic of the previous case.

Proposition 3.9 (Some High Signal Sellers Pool)**.** *If high signal sellers offer pooling prices and* $\underline{x}^h$ *is their lowest pooling offer, then* $\text{supp}(F^l) \cap \text{supp}(F^h) \cap [0, \theta_L] = [\underline{x}^h, \theta_L]$*, over which offer distributions are uniquely determined by the profit invariance conditions,*

$$\Pi(\theta_L|\theta) = \Pi(x|\theta) = (\rho + (1-\rho)(1 - F(x|\theta)))x \quad \textit{for } \theta \in \{\theta_L, \theta_H\}$$

and aggregation equations $F(\theta_L|\theta_L) = (1-\alpha)F^h(\theta_L)$*,* $F(\theta_L|\theta_H) = \alpha F^h(\theta_L)$*. Whereas* $\text{supp}(F^l) \cap \text{supp}(F^h)^c \cap [0, \theta_L] = [\underline{x}^l, \underline{x}^h]$*, over which offer distributions are uniquely determined by the profitability,*

$$\Pi(\underline{x}^l; p_L^l) = \Pi(x; p_L^l) = \left[p_L^l(\rho + (1-\rho)(1 - F(x|\theta_L))) + p_H^l(\rho + (1-\rho)(1 - F(x|\theta_H)))\right]x$$

and aggregation conditions $F(x|\theta_L) = \alpha F^l(x), F(x|\theta_H) = (1-\alpha)F^l(x)$*.*

Furthermore, given two prices $x < x' \leq \underline{x}^h$*,*

$$\Pi(x|\theta_H) < \Pi(x'|\theta_H) \quad , \quad \Pi(x|\theta_L) > \Pi(x'|\theta_L)$$

[14] See the Appendix, but the logic is similar to that which sets the profitability trend among pooling prices that are exclusively offered by low signal sellers.

Jointly, Proposition 3.5, Proposition 3.8, and Proposition 3.9 imply that offer distributions $F(x|\theta)$ are unique.

Theorem 3.1 (Unique Distributions of Offers)**.** *Equilibrium distributions of offers* $\{F(x|\theta)\}_{\theta\in\{\theta_L,\theta_H]}$*, are unique.*

These distributions determine the probability that low-valuation buyers find an acceptable offer, and thus the efficiency of trade, as well as the average price at which buyers trade, and thus consumer surplus. As a result, both of these aggregate statistics are uniquely determined.Equilibrium multiplicity is due to possible shifts in mass across seller strategies at separating prices or at pooling prices that both types offer, but only up to the point of preserving the unique offer distributions that make all these prices equally profitable. As a result, sellers' profits are also uniquely determined.

Corollary 3.1 (Efficiency and Welfare Outcome Uniqueness)**.** *Given an economy* $(\rho, \alpha, \vec{\theta}, \mathbb{P}(\theta_H))$*, the equilibrium trade probabilities and surplus of low- and high-valuation buyers, as well as, the profits of each type of seller in each type of match are invariant across equilibria.*

This allows me to conduct unambiguous comparative static analysis and to derive equilibrium strategies, in the Appendix, through a simple algorithm that weakly orders prices by the sellers' type, with the highest separating offers being made by high signal sellers, below which are any separating offers from low signal sellers or the set of pooling offers made by both types of sellers, and below which are any pooling offers that only low signal sellers extend.

3.3 Over- and Under-bidding

In these settings, there is no winner's curse: sellers' value of winning is identical conditionally or unconditionally on winning - mainly their profit-per-sale x. However, sellers do over- and under-bid, in the sense that the optimal offer ex-post is almost always different from the one they extend. Precision's profit externalities will come, in part, from exacerbating the size of these mistakes, so I will analyze them before proceeding to comparative static analysis.

Sellers' offers are optimal conditionally on their information, however, ex-post optimality requires conditioning on the complete information about the match - the buyer's valuation, the existence of competing sellers, and their type - whereas sellers' information about each of these aspects is incomplete. Consider optimal offers under three levels of nested information, where the seller knows (1) the buyer's valuation, the presence of a competing seller, and the competitor's type, (2) the buyer's valuation and the presence of a competing seller, and (3) only the buyer's valuation.

Corollary 3.2 (Optimal Prices Conditionally on Match Type)**.** *In the ordered equilibrium, conditionally on the buyer's valuation, the number of competitors, the type of the competitor,*

- *If the buyer's valuation is high, the optimal competitive price is* $x^*(\theta_H, \textit{competitive}, p_L^j \textit{ type competitor}) = \underline{x}^j$.
- *If the buyer's valuation is low, the optimal competitive price is* $x^*(\theta_L, \textit{competitive}, p_L^j \textit{ type competitor}) = \min(\underline{x}^j, \theta_L)$.

whereas, conditionally on only the buyer's valuation and the number of competitors,

- *If the buyer's valuation is high, the optimal competitive price is* $x^*(\theta_H, \textit{competitive}) = \hat{x}$.
- *If the buyer's valuation is low, the optimal competitive price is* $x^*(\theta_L, \textit{competitive}) = \underline{x}^l$.

and, conditionally on only the buyer's valuation,

- *If the buyer's valuation is high, the set of optimal price offers is* $x^*(\theta_H) = [\hat{x}, \theta_H]$.
- *If the buyer's valuation is low, the optimal price is* $x^*(\theta_L) = \underline{x}^l$.

Whereas, if the seller knows that it has no competitors and the buyer's valuation, then its optimal price offer is equal to the valuation.

The claims about optimality conditionally on only the buyer's valuation are immediate from the profitability trends that offer distributions generate[15], whereas the about optimality conditionally on also the level and type of competition are immediate from the point that lower prices in each type of sellers' support, which are less profitable in monopoly matches, are incentivized by being more profitable in competitive matches.

Since equilibrium distributions of prices are atomless, almost every offer that sellers make is suboptimal under the perfect information posterior: if they sell, their offer in that particular match was too low (over-bid buyer surplus), and inversely if they do not (under-bid buyer surplus). However, much of this regret is unavoidable, as sellers do not have information about the number of competitors, and even knowing a competitor's type, there would remain uncertainty about the price it would offer. A part of regret is linked to the buyer's valuation, though, as winning (losing) makes having been in a high (low) match more likely, and it is here that precision helps minimize regret.

Precision shapes a seller's distribution of posteriors, hence types, in two ways. First, it changes how often the seller's signal correctly classifies the buyer; in other words, how often the seller's type is relatively low or high in matches with a buyer of each valuation. Second, it changes how extreme its posterior valuation beliefs are on each signal; in other words, how large or small its type is. Together, these bring the expected sales sellers matches with each type of buyer closer to the perfect information forecast $\mathbb{P}(\text{sale at price } x|\theta)$, which allows them to improve their profitability by trading off sales with mark-ups more precisely.

3.4 Comparative Statics

With a clear understanding of the equilibrium structure, I can consider the effects of competition and precision on prices, trade efficiency, and the level/distribution of aggregate surplus between and within each side of the market.

Buyer welfare is parsed into the average utility that buyers of each valuation obtain,,

$$\mathscr{W}_b(\theta) = \rho\mathbb{E}[u(1, \min(\theta, x_i); \theta)] + (1-\rho)\mathbb{E}[\max_{i\in\{1,2\}} u(1, \min(\theta, x_i); \theta)] \tag{3.13}$$

where x_i are iid draws from the equilibrium distributions of offers $F(x_i|\theta_i)$.

Sellers are homogeneous with ex-ante profits,

$$\mathscr{M}(\tilde{\rho}) = \tilde{\rho} + 2(1-\tilde{\rho}) \tag{3.14}$$

$$\Pi = \Big[\underbrace{\mathscr{M}(\tilde{\rho})}_{\text{number of matches}} \quad \underbrace{\mathbb{P}(\text{sale at price } x, j)}_{\text{probability of selling}} \quad \underbrace{x}_{\text{profits-per-sale}} \Big] \tag{3.15}$$

where the expectation is over the events that a seller observes each signal $j \in \{l, h\}$ in a match with a buyer of each valiation $\theta_i \in \{\theta_L, \theta_H\}$ and offers a price $x \sim F^j$.

[15]Maintaining profits in each type of match over separating or overlapping pooling prices, and increasing (decreasing) profits in low- (high-) valuation matches of lower prices that only low signal sellers offer.

Lastly, because inefficient trade occurs solely from low-valuation buyers being priced out, efficiency is measured by the probability that a low-valuation buyer trades,

$$\mathcal{Q} = \quad \rho F(\theta_L|\theta_L) + (1-\rho) P_{x_1,x_2 \sim F(\cdot|\theta_L)}(\min(x_1, x_2) \leq \theta_L) \tag{3.16}$$

Less interestingly, I find that competition increases efficiency and buyer surplus. Sellers, anticipating "more" competitors in the average match, offer more attractive prices to sustain enough sales. Since sellers decrease prices across the board, all buyers obtain greater surplus, and low-valuation buyers find more acceptable offers, thus boosting trade efficiency and aggregate surplus.

More interestingly, I find that precision has nuanced effects, which closely depend on the level of adverse selection faced by sellers. The severity of adverse selection is determined by demand-side properties (a) valuation heterogeneity, $\theta_H - \theta_L$, and (b) proportion of high-valuation buyers. When adverse selection is severe, imprecisely informed sellers prioritize markups (over sales), and trade increases alongside precision, which decreases the probability of misclassification and increases the share of pooling offers upon a low signal much more than the share of separating offers upon a high signal. It increases low-valuation buyer surplus and, when trade increases strongly upon low signals, even that of high-valuation buyers. When adverse selection is mild, however, imprecisely informed sellers prioritize sales (over markups), and precision can increase the share of separating offers upon a high signal much more than the share of pooling offers upon a low signal to such a degree that it decreases trade and the surplus of all buyers. For sellers, precision is always individually beneficial, but it changes the nature of competition, so it has profit externalities for peers. This is why when all sellers obtain more precise signals[16], their profits will sometimes decrease. The settings most prone to this large negative profit externality are exactly the ones where precision is most beneficial for buyers: those where adverse selection is severe and the level of competition is high.

3.4.1 Competition

When buyers obtain more offers (lower ρ), sellers expect them to have more alternatives, particularly more alternatives below any price. Given a fixed set of strategies, this reduces expected sales at any price. Sellers respond by lowering prices to restore some sales, and since strategies are mixed, this is done by shifting mass towards lower prices. As such, the general equilibrium effect on sellers' strategies reinforces the partial equilibrium effect on the number of offers, both increasing buyer surplus and trade.

I take a closer look at this dynamic in the more relevant set of economies, those with some separating offers, and thus where predictive skill has a role. The highest overall price $\overline{x}$ does not depend on the level of competition - only on the posterior of sellers who observe a high signal (Proposition 3.3). However, the probability of prices below is closely related to the level of competition. Inspecting the distributions of offers in each type of match,

$$F(x|\theta;\rho_2) < F(x|\theta;\rho_1) \quad , \rho_1 < \rho_2$$

In other words, when there is more competition, sellers place more mass on lower prices in both types of matches (first-order stochastic dominance relation). As a result, both types of buyers obtain greater surplus, and the additional mass on pooling prices in low-valuation matches also increases trade. However, sellers' ex-ante profits decrease. This is intuitive, but to understand exactly why,

[16] Seller homogeneity constrains me to these precision counterfactuals.

recall that the sales a matched seller expects from an offer of x are,

$$\mathbb{P}(\text{sale at a price } x|j) = \rho(p_L^j \mathbb{1}(x \leq \theta_L) + p_H^j) + (1-\rho)\left(p_L^j(1-F(x|\theta_L))\mathbb{1}(x \leq \theta_L) + p_H^j(1-F(x|\theta_H))\right)$$

These decrease both because competitive matches are more likely (ρ and $1-\rho$ terms) and because the distribution of competitor bids shifts mass towards lower prices ($F(x|\theta)$ terms), making any offer is less profitable in any match. It is true that there exists a compensating effect, as competition allows sellers to match with more buyers (larger $M(\tilde{\rho}) = \tilde{\rho} + 2(1-\tilde{\rho})$), but this is insufficient. In fact, it does not affect that change of ex-ante profits, because the average number of matches also appears in the denominator of sellers' conditional probabilities of being in a monopoly or competitive match, $\rho = \frac{\tilde{\rho}}{\mathscr{M}(\tilde{\rho})}$ and $1-\rho = \frac{2(1-\tilde{\rho})}{\mathscr{M}(\tilde{\rho})}$, so it cancels out,

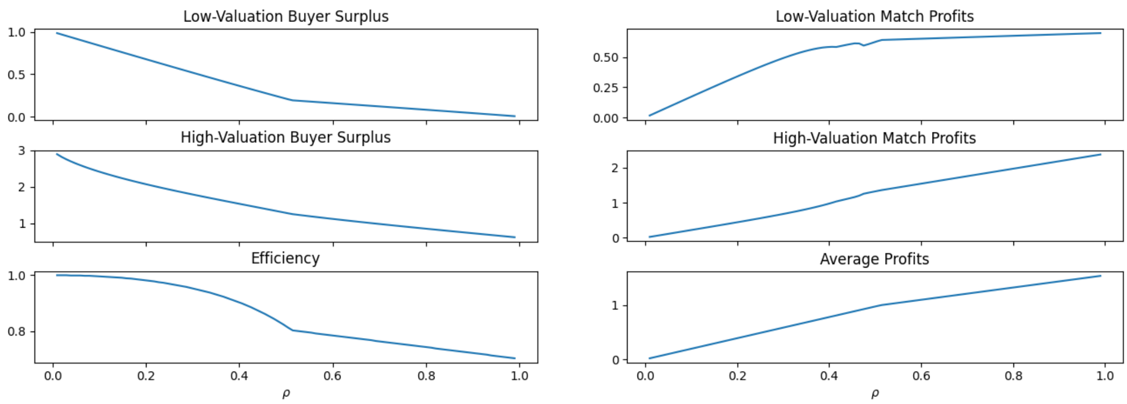
$$\begin{aligned}&\mathscr{M}(\rho)\Pi(p_L^j;\rho)\\ &= E_{x\sim F^j}\left[\tilde{\rho}(p_L^j\mathbb{1}(x \leq \theta_L) + p_H^j) + 2*(1-\tilde{\rho})\left(p_L^j\mathbb{1}(x \leq \theta_L)(1-F(x|\theta_L)) + p_H^j(1-F(x|\theta_H))\right)\right]\end{aligned}$$

As such, interim and ex-ante profits decrease with competition[17].

Proposition 3.10 (ρ Comparative Statics)**.** *Consider two economies that differ only in the level of competition $\rho_1 < \rho_2$. Then,*

- ***Distributions Strictly Increasing:*** *$F(x|\theta;\rho_2) < F(x|\theta;\rho_1)$ for $\theta \in \{\theta_L, \theta_H\}$.*
- ***Buyer Surplus Strictly Increasing:*** *$\mathscr{W}_b(\theta;\rho_2) < \mathscr{W}_b(\theta;\rho_1)$ for $\theta \in \{\theta_L, \theta_H\}$.*
- ***Trade Efficiency Increasing:*** *$\mathscr{Q}(\rho_2) \leq \mathscr{Q}(\rho_1)$.*
- ***Profits Strictly Decreasing:*** *$\Pi(\rho_1) < \Pi(\rho_2)$.*

I illustrate these effects in Figure 1.



(a) Buyer Surplus and Efficiency

(b) Seller Surplus

Figure 1: Welfare and Efficiency Effects of Competition The common parameters are $[\theta_L, \theta_H, \mathbb{P}(\theta_L)] = [1, 3, 0.5]$ for buyers and $\alpha = 0.7$ for sellers' precision.

3.4.2 Precision

Competition is traditionally dictated only by the number of offers that a buyer is expected to obtain. However, when sellers have information about preferences, its precision also affects the distribution of offers and thus the degree of competition that sellers expect for a buyer with each valuation.

[17] $\tilde{\rho}$ decreases if and only if ρ does, so $F^j(x;\rho)$ increases if and only if $\tilde{\rho}$ decreases. After canceling out the average number of matches, it follows that the terms within the expectation increase pointwise in ρ for every x.

As we know from the structure of equilibria, the more confident a seller is about being in a high-valuation match, the more sales it expects at any price, including separating prices. Precision, therefore, incentivizes sellers who observe high (low) signals to extend more (fewer) separating offers upon a high (low) signal; in other words, precision makes the offers of sellers who observe high (low) signals less (more) competitive. The net effect of greater precision among sellers, whether we consider discrete or marginal counterfactuals, is a combination of this impact on the strategy of each type of seller (low and high signal *type effect*) and its impact on the distribution of seller-types in matches with buyers of each valuation (*classification effect*).

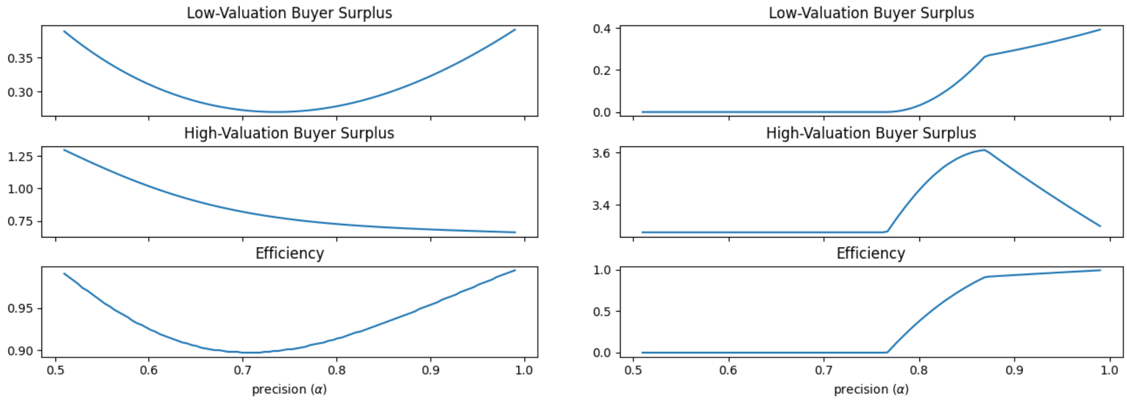
Buyer Surplus and Efficiency Pooling prices benefit high-valuation buyers, who obtain information rents, and low-valuation buyers, who experience greater competition for their business. This is why the classification effect of precision hurts high-valuation buyers and benefits low-valuation buyers, and this generally determines the sign of the relationship between precision and their surplus; however, precision's type effects (for each signal value) can dominate and reverse these relations.

Type effects are primarily determined by the heterogeneity of buyers' valuations. Imprecisely informed sellers face severe adverse selection in economies where valuations are sufficiently heterogeneous, so they prioritize high markups. Greater precision results in a small increase in the share of separating offers upon a high signal (weak high signal type effect) but a large increase in the share of pooling offers upon a low signal (strong low signal type effect). As such, precision allows low-valuation buyers to trade more often and at lower prices, with the classification effect (more sellers observe low signals in low-valuation matches) and strong low signal type effect dominating the weak high signal type effect; indeed, even high-valuation buyers benefit from precision, due to misclassification, when the low signal type effect is strong enough.

However, imprecisely informed sellers face only mild adverse selection in economies where valuations are relatively homogeneous, so they prioritize sales. Greater precision then results in a small increase in the share of pooling offers upon a low signal (weak low signal type effect) but a large increase in the share of separating offers upon a high signal (strong high signal type effect). Unless the increase in precision is large enough for sellers to rarely misclassify buyers, this technological leap then hurts all buyers and the efficiency of trade. A simple application of the product rule in the monopoly setting illustrates why,

$$F(\theta_L|\theta_L;\alpha) = \alpha F^l(\theta_L;\alpha) + (1-\alpha)F^h(\theta_L;\alpha)$$
$$\frac{\partial F}{\partial \alpha}(\theta_L|\theta_L;\alpha) = \underbrace{F^l(\theta_L;\alpha) - F^h(\theta_L;\alpha)}_{\geq 0} + \alpha\underbrace{\frac{\partial F^l}{\partial \alpha}(\theta_L;\alpha)}_{>0} + (1-\alpha)\underbrace{\frac{\partial F^h}{\partial \alpha}(\theta_L;\alpha)}_{<0}$$

Therefore, when precision is still too low (large $1-\alpha$), the increase in separating offers upon a high signal can dominate (high signal type effect), whereas if precision is higher, the decrease in misclassification (classification effect) and increase in pooling offers upon a low signal (low signal type effect) overcome it..



(a) Similar Valuations: $[\theta_L, \theta_H] = [1, 2]$ **(b)** Dissimilar Valuations: $[\theta_L, \theta_H] = [1, 10]$

Figure 2: Buyer Surplus and Efficiency Effects of Precision The common parameters are $\mathbb{P}(\theta_L) = 0.5$ for the mass of low-valuation buyers and $\rho = 0.6$ for the level of competition.

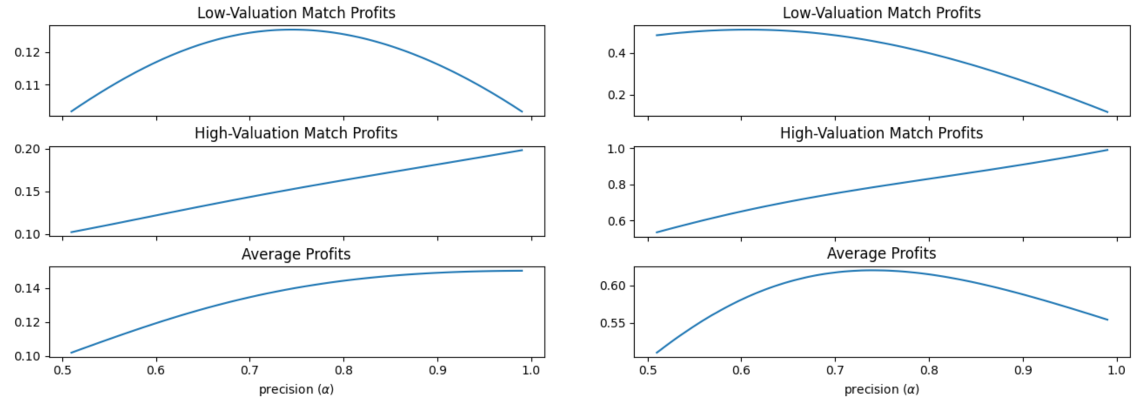
More generally, when competition is high enough, the probability of low-valuation buyers not trading has a simple form that immediately highlights how different factors impact the relationship between precision and efficiency.

Proposition 3.11. *The exists a* $0 < \tilde{\rho}(\alpha, \vec{\theta}, \mathbb{P}(\theta_H))$ *threshold level of competition, which is decreasing in* $\frac{\theta_H}{\theta_L}$ *and* $\mathbb{P}(\theta_H)$*, such that in any economy with* $\rho \leq \tilde{\rho}(\alpha, \vec{\theta}, \mathbb{P}(\theta_L))$*, such that only high signal sellers separate, some of their offers pool, and the probability that a low-valuation buyer does not trade is given by,*

$$1 - F(\theta_L|\theta_L) = \frac{\rho}{1-\rho} \frac{max\left(\frac{\mathbb{P}(\theta_H)}{\mathbb{P}(\theta_L)}\left(\frac{\theta_H}{\theta_L} - 1\right)\frac{\alpha}{1-\alpha} - 1, 0\right)}{\frac{\mathbb{P}(\theta_H)}{\mathbb{P}(\theta_L)}\frac{\alpha}{1-\alpha}\frac{\alpha}{1-\alpha} + 1} \tag{3.17}$$

Beyond a point then, competition determines the magnitude but not the sign of the relation between precision and efficiency. The sign depends on the level of precision and demand-side properties, as foreshadowed, through the severity of adverse selection. Pooling offers are subject to more severe adverse selection when its cost per instance is greater - determined by the spread of valuations - and instances are more probable - determined by the proportion of high-valuation buyers. I focus on the more nuanced demand side statistic of valuation heterogeneity throughout, but it should be understood that the proportion of high-valuation buyers has qualitatively similar effects on equilibria.

Profits Market-wide advances in precision individually benefit sellers but also generate profit externalities. When the level of competition is low (high ρ), the individual benefit dominates and profits increase with precision. However, when the level of competition is high enough, profit externalities can dominate and revert this positive relation. In particular, precision makes the offers of sellers who observe a low (high) signal more (less) competitive, and this low (high) signal type effect is strong (weak) in economies where buyers' valuations are very heterogeneous. In these economies, the net profit externality of precision is therefore negative and large, so when the individual benefit is small (such as when classification is already fairly precise), the externality dominates, causing profits to decrease with precision.



(a) Similar Valuations: $[\theta_L, \theta_H] = [1, 2]$

(b) Dissimilar Valuations: $[\theta_L, \theta_H] = [1, 10]$

Figure 3: Seller Surplus Effects of Precision The common parameters are $\mathbb{P}(\theta_L) = 0.5$ for the mass of low-valuation buyers and $\rho = 0.1$ for the level of competition.

4 Endogenous Quality Setting

Although pricing is one of the main use cases of preference information, another that is at least as important and particularly prevalent in sellers' recent data analytics applications is production - the problem of choosing what to offer to each buyer. Abstracting away from this problem has been convenient analytically and even practically negligible in situations where buyers perceive goods as highly substitutable or sellers' production is inflexible. However, to study settings where product offerings differ significantly (in quantity, quality, or probability of trade), I will incorporate product choice by allowing sellers to pick the quality of the goods they offer buyers. Although many of the core insights from exogenous product choice economies will generalize, some in exact form and others with close analogs, the significant differences lend nuance to the effects of precision on welfare and efficiency. I will also leverage the additional structure in these settings to incorporate heterogeneous seller precision in the aforementioned form of low-precision amateurs and high-precision sharks.

4.1 Seller Problem and Equilibrium Concept

The interesting cases of endogenous quality are those where the efficient quality of trade with low- and high-valuation buyers differs. This requires some cost convexity, which I introduce through a piecewise-linear cost function that kinks at the efficient qualities of trade with low- and high-valuation buyers.

Assumption 4.1 (Piecewise Linear Costs)**.** *The cost function $\phi(\cdot)$ is piecewise linear, convex, strictly increasing with*

$$\phi(q) = \begin{cases} \kappa_L q & q \leq q_L^* \\ \kappa_L q_L^* + \kappa_m q & q_L^* < q \leq q_H^* \\ \kappa_L q_L^* + \kappa_m (q_H^* - q_L^*) + \kappa_H q & q_H^* < q \end{cases}$$

where $0 < \kappa_L < \theta_L$, $\theta_L < \kappa_m < \theta_H$, and $\theta_H < \kappa_H$ and the efficient qualities of trade with a buyer of each valuation are given by,

$$q_i^* = \operatorname*{argmax}_q \theta_i q - \phi(q) \tag{4.1}$$

Piecewise linearity makes the marginal cost of quality revisions locally constant, lending considerable tractability[18]. Since sellers can choose quality and its valuation among buyers is heterogeneous, they have the ability to improve the profitability of trade with each type of buyer by going beyond single quality-price offers and instead offering menus $((q_L, x_L), (q_H, x_H))$ composed of a pair of quality-price contracts, where the contract (q_i, x_i) is intended for a buyer of valuation θ_i. When both contracts are identical, the menu is equivalent to a single contract offer, but we will see that such menus are never optimal, the first sign that the exogenous product choice assumption imposes serious economic limitations.

There are three components to the profits that a matched seller expects from a menu: (1) the profits per sale from each contract, (2) the probability that a buyer of a given valuation chooses a contract, and (3) the probabilities that the buyer has each valuation. Profits per sale from a contract, $\pi(q_i, x_i) = x_i - \phi(q_i)$, are simply the difference between the lump sum price x_i and the seller's cost of producing the quality q_i. The probability that a buyer of a given valuation chooses a contract is the probability that it has no better offers. By the Revelation Principle, it is sufficient to restrict attention to individually rational and incentive-compatible menus, so the probability that a buyer of valuation θ_i chooses the contract (q_j, x_j) is zero if $i \neq j$, and otherwise equal to the sum of the probability that the seller's offer is the only one (ρ) plus the probability that it has an inferior offer from another seller. To compute the probability that the seller's offer beats that of a competitor, I define the marginal distribution $F_i(u_i) = F(u_i \times [0, \infty]|\theta_i)$ over indirect utilities offered to θ_i valuation buyers by sellers of each type via the joint distribution of utilities in such matches,

$$F(u_L, u_H|\theta_i) = \sum_{\substack{e \in \{a,s\} \\ j \in \{l,h\}}} \mu(e) P^e(j|\theta_i) F^{e,j}(u_L, u_H) \tag{4.2}$$

where $\mu(e)$ is the mass of sellers with precision α_e, $P^e(j|\theta_i)$ is the proportion of them that would observe j signals when matched with a θ_i valuation buyer (and so that would be of type $p_L^{e,j}$), and $F^{e,j}(u_L, u_H)$ is the joint distribution over indirect utility offers implied the menus that sellers of type $p_L^{e,j}$ mix over. Therefore, the probability that a θ_i valuation buyer chooses a contract (q_i, x_i) contract is $\Psi_i(u_i) = \rho + (1-\rho)F_i(u_i)$, where $u_i = u(q_i, x_i; \theta_i)$ is the indirect utility it offers θ_i valuation buyers. Lastly, a matched seller's probability that the buyer has low and high-valuation is given by its type $p_L^{e,j}$ and the respective complementary probability $1 - p_L^{e,j}$. Since each contract (q_i, x_i) determines the seller's expected profits in a match with each type of buyer, the expected profits from the menu are therefore given by the average of these,

$$\begin{aligned} \Pi^{e,j}(q_L, x_L, q_H, x_H) &= p_L^{e,j} \Psi_L(u(q_L, x_L; \theta_L))\pi(q_L, x_L) + p_H^{e,j} \Psi_H(u(q_H, x_H; \theta_H))\pi(q_H, x_H) \\ &= \sum_{i=l,h} p_i^{e,j} \Psi_i(u(q_i, x_i; \theta_i))\pi(q_i, x_i) \end{aligned} \tag{4.3}$$

weighted by the seller's valuation beliefs.

4.2 Seller Strategies

I have so far allowed for both pooling and separating menu offers; however, as the notation suggests, only separating menus are offered in equilibrium.

Corollary 4.1 (No Pooling in Equilibrium)**.** *Equilibrium menus separate buyers of each valuation.*

[18]The main results are not dependent on this restriction.

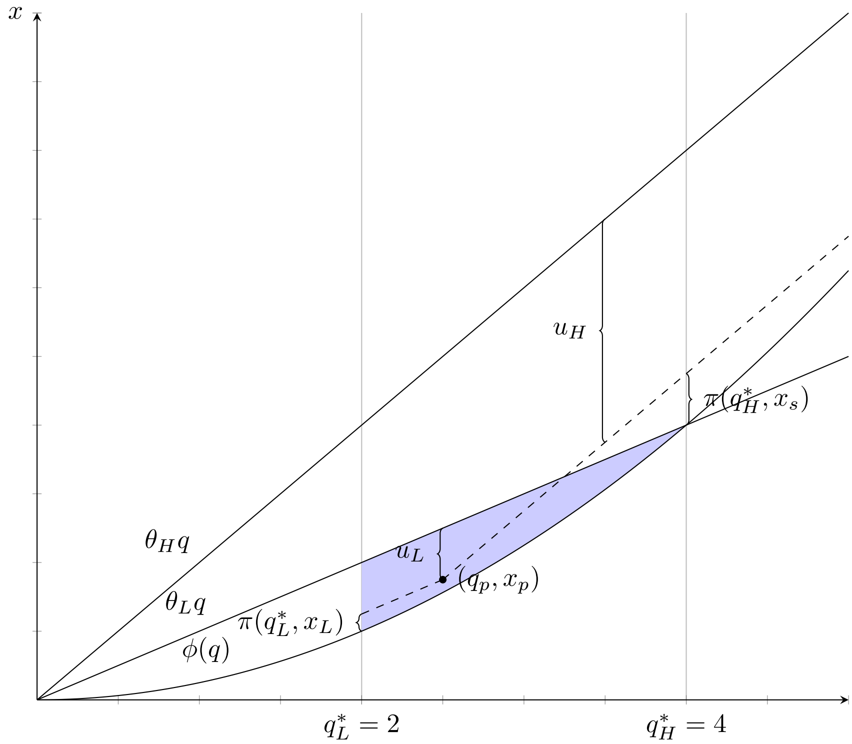


Figure 4: Example with $\theta_H = 4$, $\theta_L = 2$, $\phi(q) = \frac{1}{2}x^2$. Contracts are depicted as points in the (q, x) space. Implied utilities are given by the vertical distance of a contract's y-axis coordinate to the zero utility indifference curves of the respective θ_i valuation buyers, while profits per sale are given by the vertical distance to the seller's cost function $\phi(q)$. The blue region represents the set of pooling contracts that are individually rational for buyers of either valuation, imply non-negative profits, and are not dominated by another pooling contract.

A candidate pooling offer (q_p, x_p) lies on dashed iso-utility lines for buyers with low and high-valuation; I follow these to the right and left, respectively, until reaching the efficient qualities of trade with each type of buyer at prices $x_i = \theta_j q_i^* - u(q_p, x_p; \theta_i)$. This alternative separating bid $((q_L = q_L^*, x_L = x_p - \theta_L(q_p - q_L^*), (q_H = q_H^*, x_H = x_p + \theta(q_H^* - q_p))$ remains incentive compatible and strictly dominates the pooling offer: the same utility to buyers of either valuation (hence the same probability of winning) but strictly higher profits per sale.

Cost convexity is at the core of this result. It is the reason why a pooling contract can always be improved through a separating revision that adds quality to the high contract, at a price (above marginal costs) only high-valuation buyers are willing to pay, and reduces the quality of the low, in exchange for a discount only low-valuation buyers are interested in. I provide a simple graphical representation of this procedure in Figure 4. Beyond its economic relevance, this result is analytically convenient because it eliminates both classical threats to existence of equilibrium (as in Rothschild and Stiglitz (1976)) and a more complicated diversity of offers (as in Lester et al. (2019)).

Beyond separating buyers, equilibrium menus have quite a bit of additional structure. In particular, the quality-price terms of each contract featured in a menu are closely linked to the utility they offer each type of buyer. The forward direction is obvious since incentive constraints and individual rationality imply buyers will always choose their intended contract,

$$(IC_i) \; : \quad u(q_i, x_i; \theta_i) \geq u(q_{\neg i}, x_{\neg i}; \theta_i) \; \forall i \in \{l, h\}$$
$$(IR_i) \; : \quad u(q_i, x_i; \theta_i) \geq 0$$

so the indirect utility offered to buyers of each respective valuation by a menu is simply,

$$u_L = \theta_L q_L - x_L$$
$$u_H = \theta_H q_H - x_H$$

The backward direction follows from the conditions that profit maximality imposes on optimal offers, which generate a bijection between the indirect utility terms (u_L, u_H) and quality-price terms $((q_L, x_L), (q_H, x_H))$ of any *equilibrium* menu.

Theorem 4.1 (Converting to Indirect Utilities). *Consider an equilibrium menu* $((q_L, x_L), (q_H, x_H))$ *with associated indirect utilities* (u_L, u_H)*. Qualities are then given by,*

$$q_L(u_L, u_H) = \begin{cases} \frac{u_H - u_L}{\Delta\theta} & u_H - u_L < q_L^* \Delta\theta \\ q_L^* & u_H - u_L \geq q_L^* \Delta\theta \end{cases} \qquad q_H(u_L, u_H) = \begin{cases} \frac{u_H - u_L}{\Delta\theta} & u_H - u_L > q_H^* \Delta\theta \\ q_H^* & u_H - u_L \leq q_H^* \Delta\theta \end{cases} \tag{4.4}$$

and prices by $x_i = \theta_i q_i - u_i$*, where* $\Delta\theta = \theta_H - \theta_L$*.*

Therefore, when an incentive constraint binds at an optimal menu, the difference in utilities $u_H - u_L$ uniquely determines the qualities offered in each contract, while the level of these utilities then uniquely determines their respective prices. Whereas, when both incentive constraints are slack, it is optimal to offer efficient qualities to each type of buyer, which is why I will refer to these menus as *dually efficient*, and prices follow uniquely in the same fashion.

This result connecting the quality-price form of menus to their associated indirect utilities is standard - referred to as the *parametric-utility approach* (Rochet and Stole (2006)) - and originates from the fact that seller surplus (profits per sale) equals the gains from trade $S_i(q) = \theta_i q - \phi(q)$ net of buyer surplus,

$$\pi(q_i, x_H) = x_i - \phi(q_i) = (x_i - \theta_i q_i) + (\theta_i q_i - \phi(q_i)) = S_i(q_i) - u(q_i, x_i; \theta_i)$$

Profits thus increase both from minimizing buyer surplus *and* maximizing trade efficiency. As such, consider the optimal way to offer a pair of utilities (u_L, u_H). If it is feasible to do so with a dually efficient menu, while respecting incentive constraints, then this is optimal, for it generates more social surplus. If it is not possible and a buyer's incentive constraint would be violated by such an offer, then a problematic IC_H (IC_L) constraint is corrected most profitably by under-providing (over-providing) quality in the low (high) contract. In these cases, a valuation θ_i buyer - whose incentive constraint binds at the optimal menu that offers (u_L, u_H) - is indifferent between their contract and the one intended for a buyer with the opposite valuation, so $u_H - u_L = q_{\neg i}\Delta\theta$.

Since optimal offers can be expressed in terms of indirect utilities, it is convenient to recast the strategy of each type of seller $p_L^{e,j}$ as a distribution $F^{e,j}(u_L, u_H)$ over pairs of utility offers (u_L, u_H). This not only reduces the dimension of the space of offers, but also directly links them to buyer surplus. In this notation, the profits that a type $p_L^{e,j}$ seller expects from an offer (u_L, u_H) are,

$$\Pi_i(u_L, u_H) = \Psi_i(u_i)\pi_i(u_L, u_H)$$
$$\Pi^{e,j}(u_L, u_H) = \sum_{i=l,h} p_i^{e,j} \Pi_i(u_L, u_H)$$

Therefore, the *level* of indirect utilities (u_L, u_H) determines both the probability of winning in contested matches $F_i(u_i)$ and the buyer surplus part of profits per sale $\pi(u_L, u_H) = S(u_L, u_H) - u_i$, while the *difference* in utilities, $u_H - u_L$, determines the social surplus of profits per sale. Given the sample space $\Omega = [0, S_L^*] \times [0, S_H^*]$, Borel σ-algebra, and set of countably additive probability measures $\mathscr{P}$ over it, this concise specification of the seller's problem allows me to introduce the equilibrium concept.

Definition 4.1 (Bayes-Nash Equilibrium)**.** *An equilibrium is a vector of strategies* $\{F^{e,j}\}_{\substack{e\in\{a,s\} \\ j\in\{l,h\}}}$*, for each type of seller, such that,*

$$\operatorname{supp}(F^{e,j}) \subseteq \underset{(u_L,u_H)}{\operatorname{argmax}} \, \Pi^{e,j}(u_L, u_H) \quad \forall\ (e,j) \in \{a,s\} \times \{l,h\}$$

4.3 Equilibrium Structure

4.3.1 Corner Cases: Monopoly and Perfect Competition

We build intuition starting at the corner cases of competition, as in economies with exogenous quality. The screening problem of a monopolist is well understood from Mussa and Rosen (1978). A type $p_L^{e,j}$ monopolist thus solves

$$\max_{u_L,u_H\geq 0} p_L^{e,j}(S_L(u_L,u_H) - u_L) + p_H^{e,j}(S_H(u_L,u_H) - u_H)$$

where incentive compatibility is implicit in the functional form of social surplus terms. We can ignore menus with (1) $u_H - u_L > q_L^* \Delta\theta$ or (2) $u_L > 0$, as they would not entail greater efficiency of trade and strictly lower profits per sale from buyers of at least one valuation than the offer $(u_L, u_H) = (0, q_L^* \Delta\theta)$. The incentive constraint of high-valuation buyers, therefore, binds at an optimal menu, and the optimal quality in the low contract. The only trade-off for a monopolist is between the efficiency of trade with low-valuation buyers and the share of the efficient social surplus S_H^* that it captures in trade with high-valuation buyers. Sellers have piecewise linear costs that give rise to a bang-bang monopolist outcome, with a fully efficient or inefficient offer, depending on whether the seller's type $p_L^{e,j}$ is beyond a threshold belief p_L^* determined by the marginal benefit of increasing the indirect utility offered to high-valuation buyers,

$$p_L^* \underbrace{\frac{\theta_L - \kappa_L}{\Delta\theta}}_{\text{efficiency gain}} - \underbrace{p_H^*}_{\text{rent loss}} \tag{4.5}$$

The left-hand term is the marginal gain in efficiency made possible by increasing the indirect utility offered to high-valuation buyers. This relaxes their incentive constraint and allows the seller to provide a more efficient quality to low-valuation buyers, increasing the gains from trade with them and thus the profitability of these sales. The right-hand term represents the rents surrendered by making a more generous offer to high-valuation buyers; in other words, of offering the efficient high quality at a strictly lower price.

Proposition 4.1 (Monopoly)**.** *If $\rho = 1$, a seller of type $p_L^{e,j}$ offers a menu,*

$$(u_L, u_H) = \begin{cases} (0,0) & \text{if } p_L^{e,j} < p_L^* \\ (0, q_L^*\Delta\theta) & \text{if } p_L^{e,j} \geq p_L^* \end{cases} \tag{4.6}$$

for the threshold probability,

$$p_L^* = \frac{\theta_H - \theta_L}{\theta_H - \kappa_L} \tag{4.7}$$

We observe properties familiar from economies where quality was exogenous, mainly the link between precision and offer dispersion, the monotonicity of efficiency and generosity with a seller's type, the nonmonotonicity between precision and welfare aggregates, and the dependence of these relations on preference heterogeneity $\Delta\theta$, which explicitly determines the size of the efficiency gain $\frac{\theta_L - \kappa_L}{\Delta\theta}$ from relaxing high-valuation buyers' incentive constraint. In detail, when buyers' valuations are sufficiently homogeneous, imprecisely informed monopolists always trade efficiently ($p_L^{e,j} > p_L^*$ for both signals $j \in \{l, h\}$), so additional precision can decrease the efficiency of trade ($p_L^{e,h} < p_L^*$ for α_e large) and hurt high-valuation buyers (low-valuation buyers always get zero surplus). However, when buyers' valuations are sufficiently heterogeneous, imprecisely informed monopolists ration low-valuation buyers upon the high and even, sometimes, the low signal, so precision increases the efficiency of trade and can also allow high-valuation buyers to obtain some information rents from misclassification. These ordering, efficiency, and welfare relations extend when competition is imperfect, with the usual caveat that competition allows low-valuation buyers to obtain some of the gains from trade and thus benefit from precision, so long as it is not too efficiency-reducing.

The other corner where sellers are guaranteed to bid against another seller in every match ($\rho = 0$) is also well understood, as each match again becomes a setting of Bertrand price competition. Therefore, sellers make dually efficient offers and buyers capture all gains from trade.

Proposition 4.2 (Perfect Competition)**.** *If $\rho = 0$, the unique Bayes-Nash equilibrium involves almost every seller offering $((q_L, x_L), (q_H, x_H)) = ((q_L^*, \phi(q_L^*)), (q_H^*, \phi(q_H^*)))$ with probability 1.*

Second-degree price discrimination does not change the point that predictive skill is of little use under perfect competition. Competition also promotes efficiency and consumer surplus when it is imperfect, but it has different effects on how much sharks and amateurs use their predictive skill, as the latter's offers are dually efficient and thus invariant to their signal in more settings. Lastly, note that the combination of imperfect competition and heterogeneous seller precision allows me to isolate its profit externalities by fixing the precision of one group of sellers (for example, amateurs) and varying the precision of the other (for example, sharks). I find that sellers become more profitable through their own predictive skill, and sometimes also that of competitors.

4.3.2 Interior Cases: Imperfect Competition

In this section, I discuss some general properties satisfied by the menus of a candidate equilibrium. Each has intuitive appeal, either from an economic standpoint or because of the mathematical tractability that they impart. By leveraging these in conjunction with the optimality conditions of sellers' problems, I can solve for this equilibrium analytically and obtain a complete characterization. In the Appendix, I show that these properties hold in any equilibrium where at least some offer rations low-valuation buyers.

There are three main areas that benefit from additional structure: (1) the distributions over indirect utilities offered by sellers in low- and high-valuation matches, F_i, (2) the relationship of incentive compatibility constraints to the generosity (indirect utility) of menus, and (3) the relationship between the generosity of a menu with the type of seller who offers it.

The distributions of indirect utilities F_i offered in each match are weighted averages of each type of seller's mixed strategy, so they inherit the properties of these strategies.

Claim 4.1. *The equilibrium marginal distributions over indirect utilities F_i for $i \in \{l, h\}$,*

1. *are atomless.*

2. *have a connected support of low utility offerings $\Upsilon_L = [\underline{u}_L, \overline{u}_L]$.*

3. *have a support of high utility offerings* $\Upsilon_H = \cup_{\substack{e \in \{a,s\},\\ j \in \{l,h\}}} [\underline{u}_H^{e,j}, \overline{u}_H^{e,j}]$, *made up of the contiguous bids by type* $p_L^{e,j}$ *sellers.*
4. *are continuously differentiable with densities* f_i *in the interior and one-sided derivatives at the boundaries.*

By avoiding atoms and gaps in the supports of each type of seller's mixture, erratic behavior is curtailed. It is never optimal to bunch up and compete at a single point in the space of utility offers, as one might see in the corner cases of monopoly and Bertrand, nor are there discontinuous jumps in generosity among sellers of the same type. While continuous densities allow me to consider marginal incentives, which helps convey economic intuition and solve for the equilibrium via a standard differential system.

As in the closely related work of Lester et al. (2019) and Garret et al. (2019), a property called *ordering*, which refers to an equilibrium where ranking menus by the utility offered to low- or high-valuation buyers is identical, lends a great degree of tractability. This property turns out to be necessary in any equilibrium where the incentive constraint of low-valuation buyers is slack in every menu that is offered, and a sufficient condition for this is the assumption that sellers cannot profitably trade the efficient high quality with low-valuation buyers.

Assumption 4.2 (No IC_L Cost Condition)**.** *The piecewise linear cost function* $\phi(\cdot)$ *is such that,*

$$\theta_L q_H^* \leq \kappa_L q_L^* + \kappa_m (q_H^* - q_L^*) \tag{4.8}$$

Claim 4.2. *Low-valuation buyers' incentive constraint never binds in equilibrium menus.*

Eliminating the possibility of a binding low-valuation buyer constraint means that high-valuation buyers always obtain their efficient quality (q_H^*), but low-valuation buyers are rationed ($q_L < q_L^*$) whenever the offer intended for them is part of a menu in which high-valuation buyers' incentive constraint binds.

I will focus on equilibria in which offers are (a) ordered by their generosity, but also (b) monotone in the seller's type, meaning that generosity weakly increases with the seller's belief of being matched with a low-valuation buyer $p_L^{e,j}$, as in the ordered equilibria of economies with exogenous quality.

Claim 4.3. *Given two equilibrium menus* (u_L, u_H) *and* $(\tilde{u}_L, \tilde{u}_H)$ *offered by sellers of respective types* p_L *and* $\tilde{p}_L$ *with* $u_i > \tilde{u}_i$ *for some* $i \in \{l, h\}$,

1. *It is also true that* $u_{\neg i} \geq \tilde{u}_{\neg i}$ *and strictly so if both menus are offered by sellers of the same type.*
2. *The gap between low and high utilities* $u_H - u_L$ *increases strictly with generosity.*

Due to Theorem 4.1, the efficiency of any menu is directly pinned down by the difference in its utility offers $u_H - u_L$, and since low-valuation buyers' incentive constraint does not bind in equilibrium menus, a larger difference in utility offers only improves efficiency - weakly raising the quality of low trade towards q_L^*. Orderedness in generosity thus follows from the complementarity between relaxing the incentive compatibility constraint of a high-valuation buyer (through a more generous u_H term) - which permits more profitable sales to low-valuation buyers - and increasing low sales (through a more generous u_L term). In equilibrium, the first effect dominates - sellers face stronger incentives to increase the generosity of their high- versus low-valuation offer - so that the difference in utilities $u_H - u_L$ and, consequently, efficiency increases weakly with generosity. This upward efficiency progression generates a natural grouping of menus, with the least generous menus also being the least efficient, but above a generosity threshold, becoming dually efficient.

Equilibrium profits conditionally on matching with a low (high) buyer, then provide the incentives that order the offers of sellers with different interim valuation beliefs (types). These give rise to the second ordering relation, which links a seller's type to the generosity of its offers. In particular, the profitability of menus in low- (high-) valuation matches increases (decrease) with their generosity and lead sellers who place more posterior weight on low-valuation matches to extend more generous offers.

Claim 4.4. *Given two equilibrium menus (u_L, u_H) and $(\tilde{u}_L, \tilde{u}_H)$ offered by sellers of respective types p_L and $\tilde{p}_L$ with $u_i > \tilde{u}_i$ for some $i \in \{l, h\}$,*

1. *Profitability conditionally on matching with a buyer of low- (high-) valuation is increasing (decreasing) in generosity,*

$$\Pi_L(u_L, u_H) \geq \Pi_L(\tilde{u}_L, \tilde{u}_H) \text{ and } \Pi_H(u_L, u_H) \leq \Pi_H(\tilde{u}_L, \tilde{u}_H)$$

with strict inequalities if high-valuation buyers' incentive constraint binds at $(\tilde{u}_L, \tilde{u}_H)$

2. *Generosity is weakly increasing in the seller's belief of being matched with a low-valuation buyer, $p_L > \tilde{p}_L$, and strictly so if high-valuation buyers' incentive constraint binds at $(\tilde{u}_L, \tilde{u}_H)$.*

3. *Each support $\Upsilon_i = [\underline{u}_i, \overline{u}_i]$ is such that $\overline{u}_i \leq S_i^*$. Further, there exists a $u_i^{de} \in [\underline{u}_i, \overline{u}_i]$ such that all $u_i < u_i^{de}$ are in IC_H binding menus and all $u_i \geq u_i^{de}$ are dually efficient.*

The origin of this conditional profit monotonicity is clear when we consider the first-order condition satisfied by the bids of sellers offering menus at which high-valuation buyers' incentive constraint binds,

$$0 = p_H^{e,j} \frac{\partial \Pi_H}{\partial u_H}(u_H, u_L) + p_L^{e,j} \underbrace{\frac{\partial \Pi_L}{\partial u_H}(u_H, u_L)}_{>0}$$

$$0 = p_L^{e,j} \frac{\partial \Pi_L}{\partial u_L}(u_H, u_L)$$

When high-valuation buyers' incentive constraint binds at a menu, greater generosity toward them (larger u_H) relaxes the constraint and makes low trade more profitable, so the term $\frac{\partial \Pi_L}{\partial u_H}(u_H, u_L)$ is strictly positive. As such, the profits from high-valuation sales of constrained menus (Π_H) are locally decreasing in generosity towards high-valuation buyers (u_H) but invariant in that towards the low (u_L), while their profits from low-valuation sales (Π_L) are increasing generosity towards high-valuation buyers and at a local maximum with respect to low-valuation buyer generosity. When we consider the set of constrained menus, the least generous are offered by the sellers who are comparatively more concerned about the profitability in high-valuation matches, mainly those sellers of the lowest type $p_L^{e,j}$. The efficiency-generosity relation then places these constrained menus below any dually efficient ones, so sellers who are sufficiently concerned about low-valuation match profits offer the latter (dually efficient) menus. Incentive constraints are slack among dually efficient menus, however, so marginal changes in the generosity of the contract offered to a buyer of either type does not affect the profitability of the paired contract (by Theorem 4.1). The term $\frac{\partial \Pi_L}{\partial u_H}(u_H, u_L)$ is, therefore, equal to zero among dually efficient offers, and these satisfy the pair of equations,

$$0 = \frac{\partial \Pi_H}{\partial u_H}(u_H, u_L)$$
$$0 = \frac{\partial \Pi_L}{\partial u_L}(u_H, u_L)$$

implying that dually efficient offers are equally profitable and that all sellers - irrespective of their type - are indifferent between them. The most that we can necessarily say about the relation between a seller's type and dually efficient menus is that the type must be high enough for the seller to offer one. Any additional relation between sellers' type and the generosity of dually efficient menus is not just unnecessary but unappealing because it would allow precision to shape the share of surplus that buyers of each valuation obtain, purely due to a particular equilibrium selection rule rather than an essential and ubiquitous effect. Instead, I consider the unique equilibrium where any seller has the same distribution of dually efficient offers conditionally on making any.

Ordering in economies with endogenous quality, therefore, differs from that in economies with exogenous quality. Principally, in the latter, the mass that each type of seller places on inefficient offers (price larger than θ_L) can be arbitrarily shifted while preserving the aggregate offer distribution, while among efficient offers (price lower than θ_L) the lowest ones must be offered by sellers of the largest type (unless all offers are efficient). Whereas in economies with endogenous quality, the opposite is true, as inefficient offers must be strictly ranked by the type of seller who offers it, whereas efficient offers are not.

I close by connecting these points about profitability, generosity, and a seller's type to the benefits of precision. With two information structures (levels of precision), each of which features a random variable (signal) that can take two possible values (low or high), there are four types of sellers at the interim stage, corresponding to each precision and signal combination. And, sellers with the highest precision have the highest conviction about their signals and the most extreme beliefs on each signal,

$$p_L^{s,h} < p_L^{a,h} < p_L^{a,l} < p_L^{s,l} \tag{4.9}$$

As such, in an equilibrium where at least some inefficient menus are offered, the least generous menus, towards a buyer of either valuation, are offered by sharks who observe high signals, while the most generous are offered by sharks who observe low signals. Amateur offers have intermediate generosity and only overlap with those of sharks if they are dually efficient. This means that in low- (high-) valuation matches, a greater share of sharks (than amateurs) observe signals that cue them to place their offers at the top (bottom) of the generosity distribution, where the most profitable low- (high-) valuation match menus are found, and makes them more profitable than amateurs in the average match - the benefit of predictive skill.

4.4 Comparative Statics

I will perform an analogous comparative static analysis to Section 3.4, highlighting points of commonality and departure. The aggregate statistics for welfare and trade efficiency are similar after adjusting for the fact that strategies are expressed in the space of indirect utility offers and that sellers are heterogeneous. Recycling notation, the welfare of each type of buyer follows as,

$$\mathscr{W}_b(\theta) = \rho \mathbb{E}[u_i] + (1-\rho)\mathbb{E}[\max(u_i, \tilde{u}_i)] \tag{4.10}$$

where u_i are iid draws from the cmarginals $F_i(u_i)$. Profits of sellers with precision α_e are given by,

$$\Pi^e = E\Big[\underbrace{\mathscr{M}}_{\text{number of matches}} \underbrace{\Psi_i(u_i)}_{\text{probability of selling}} \underbrace{S_i(u_L, u_H) - u_i}_{\text{profits-per-sale}} \Big| e \Big] \tag{4.11}$$

where I also average over the probabilities that they observe each signal $j \in \{l, h\}$ in matches with a buyer of each valuation $\theta_i \in \{\theta_L, \theta_H\}$, and that they mix over menus as per $F^{e,j}$ upon each signal. Lastly, in the case of efficiency, high-valuation buyers always receive efficient offers, but the low may get rationed, so I measure the efficiency of trade by the average quality offered to the latter,

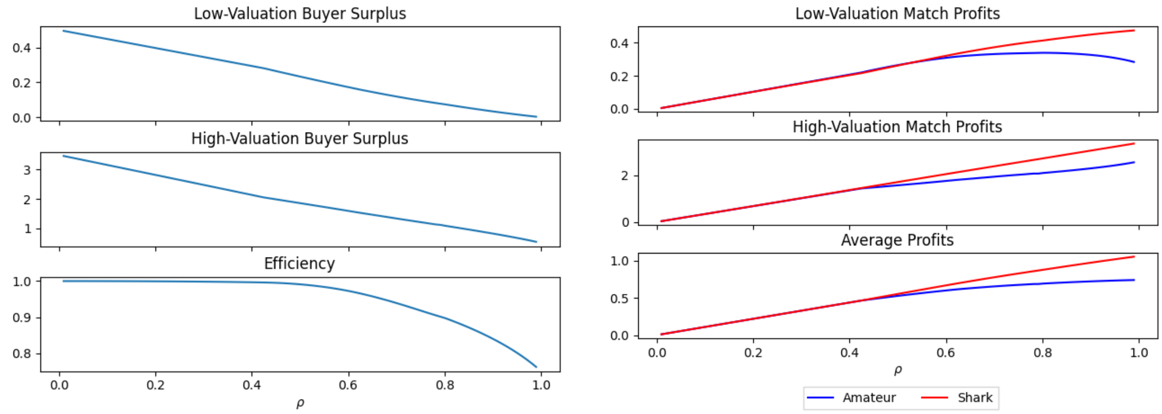
$$\rho\mathbb{E}[q_L|\theta_L] + (1-\rho)\mathbb{E}[\max(q_L, q_L')|\theta_L] \tag{4.12}$$

where I rely on the equilibrium property that generosity and efficiency are positively correlated, so buyers always select the most efficient offer.

The qualitative effects of competition and precision on efficiency and buyer surplus are directionally similar in similar environments, but we look elsewhere to track efficiency. Whereas pricing uniquely impacts trade efficiency at the extensive margin - determining whether a buyer trades or not by finding any acceptable prices - when quality is exogenous, removing this barrier allows information to shape the level of trade at the intensive margin - determining the quality of trade a buyer obtains. Stronger still, the optimality of screening menus allows almost every buyer to trade *some* amount, so that *all* action is at the intensive margin - a theoretical insight that informs empirical analysis. Seller heterogeneity introduces more significant differences. For one, it allows me to cleanly analyze precision's profit externalities and find that precision growth at the frontier (shark precision) is broadly beneficial for *all* sellers, even laggards, whereas precision growth among laggards generally only benefits them and hurts more leaders. Second, it gives rise to the ordering of interim types (4.9) that orders offers and which, in turn, implies that in sufficiently competitive economies, amateurs do not use their predictive skill, since their offers on either signal are dually efficient and equally distributed. In other words, competition neutralizes the predictive skill of amateurs, when quality is endogenous.

4.4.1 Competition

The aggregate effects of competition are similar when information also orients production - increasing trade efficiency, buyer surplus, and decreasing seller surplus - but the mechanism responsible for them differs. In particular, competition increases the sales gains from generosity, so sellers offer more generous menus, and complementarity between the utility offered to each type of buyer then links both their joint progression and the quality that is offered to low-valuation buyers.

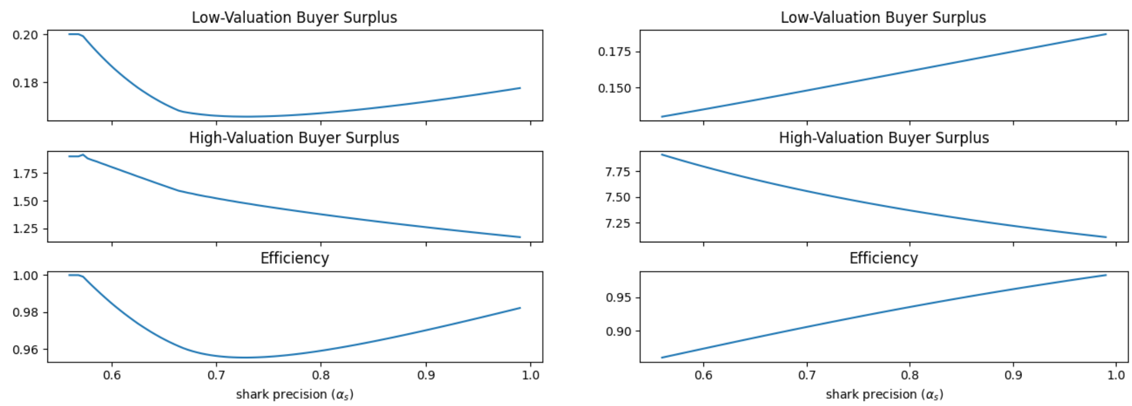


(a) Buyer Surplus and Efficiency **(b) Seller Surplus**

Figure 5: Buyer Surplus and Efficiency Effects of Competition The common parameters are $[\theta_L, \theta_H, \mathbb{P}(\theta_L)] = [1, 3, 0.8]$ for buyers, $[\kappa_L, \kappa_m, q_L^*, q_H^*] = [0.5, 2, 1, 2]$ for sellers' cost functions, $[\alpha_a, \alpha_s] = [0.55, 0.95]$ for sellers' precision, and $\mathbb{P}(a) = 0.5$ for proportion of amateurs.

4.4.2 Precision

The qualitative effects of precision on the efficiency of trade and buyer surplus are similar as well, with valuation heterogeneity largely determining precision's effects: in economies where buyers' preferences are similar, precision can decrease trade efficiency and hurt all buyers, whereas in economies where buyers' preferences are dissimilar, precision increases trade efficiency and low-valuation buyer surplus, but generally[19] hurts high-valuation buyers. I illustrate these points in Figure 6 by tracking the response of buyer surplus and trade efficiency to changes in shark precision.



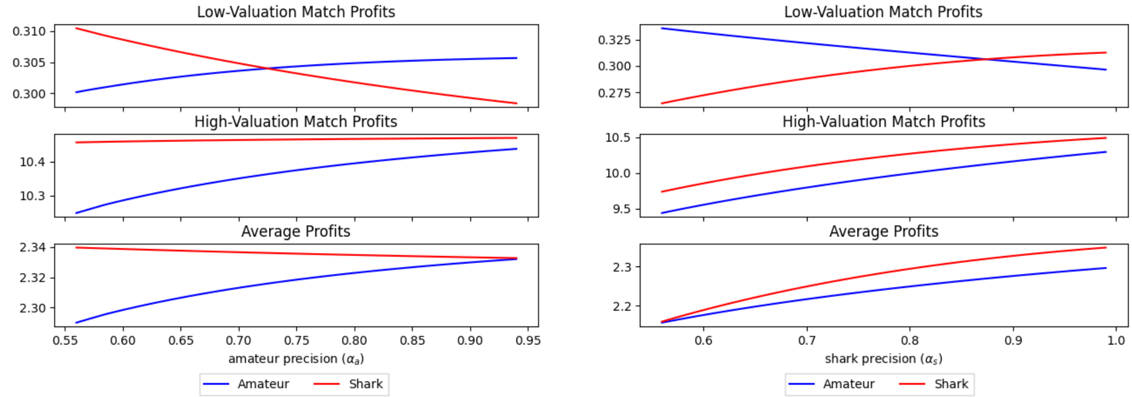
(a) Similar Valuations: $[\theta_L, \theta_H] = [1, 2.5]$ **(b)** Dissimilar Valuations: $[\theta_L, \theta_H] = [1, 10]$

Figure 6: Buyer Surplus and Efficiency Effects of Precision Half of the sellers are sharks and I increase their precision. The common parameters are $\mathbb{P}(\theta_L) = 0.8$ for the mass of low-valuation buyers, $[\kappa_L, \kappa_m, q_L^*, q_H^*] = [0.5, 2, 1, 2]$ for sellers' cost functions, $\alpha_a = 0.55$ for amateur precision, and $\rho = 0.6$ for the level of competition.

When it comes to sellers, the monotonicity of offers with seller type is fundamental. Inefficient offers from amateurs only beat those of sharks who observe high signals and the profitability of dually efficient offers is only determined by the generosity of inefficient offers below, so shark precision sets off

[19] There are still cases where additional precision benefits high-valuation buyers, by encouraging generosity upon a low signal (low signal type effect).

a chain of downward generosity revisions that benefit both them and amateurs: the inefficient offers of high signal sharks become less competitive (generous) when their information is more precise (lower $p_L^{s,h}$ type under a larger α_s), which makes any of the following inefficient offers of amateurs above less competitive/generous, which makes any of the following inefficient offers of low signal sharks above less competitive/generous, ultimately also making any of the following dually efficient offers above less competitive/generous.



(a) Amateur Precision Change, Shark Precision $\alpha_s = 0.95$

(b) Shark Precision Change, Amateur Precision $\alpha_a = 0.55$

Figure 7: Seller Surplus Effects of Precision The common parameters are $[\theta_L, \theta_H, \mathbb{P}(\theta_L)] = [1, 10, 0.8]$ for buyers, $[\kappa_L, \kappa_m, q_L^*, q_H^*, \mathbb{P}(a)] = [0.5, 2, 1, 2, 0.5]$ for sellers, and $\rho = 0.6$ for the level of competition.

On the other hand, sharks generally suffer from amateur expertise. Sharks who observe a high signal always lose against amateurs, so their profitability is invariant to amateur precision, but sharks who observe a low signal always beat amateurs, so they are exposed to the effect of precision on amateurs who observe high and who low signals. High signal amateurs' offers become less competitive/generous under greater precision, particularly affecting their utility offers to high-valuation buyers, and this chains up the distribution, so sharks' sales to high-valuation buyers generally become more profitable. However, low signal amateurs become willing to bid more aggressively for any amount of low-valuation buyer trade, so they strongly increase the utility offered to these buyers and decrease the profitability of sharks' sales to them. It is the latter effect that generally dominates; the exceptions, where all sellers benefit from amateur precision at the margin, take place in settings where buyers have very similar preferences and amateur precision is relatively low. I illustrate the general profitability trends in Figure 7 and the more exceptional one in Figure 8.

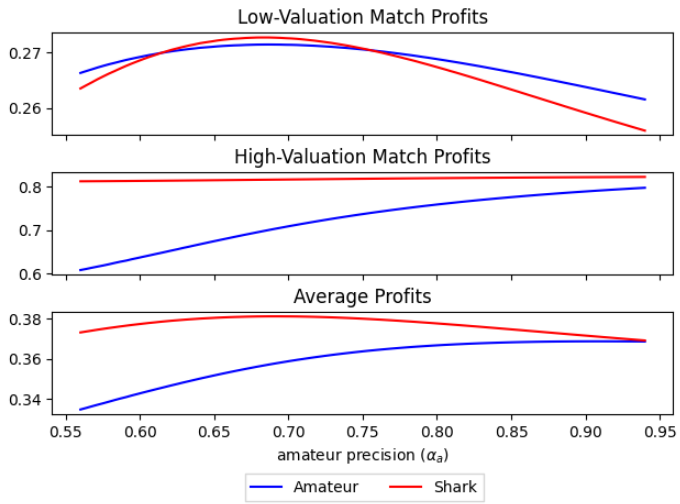


Figure 8: All Sellers Benefit from Amateur Precision The parameters are $[\theta_L, \theta_H, \mathbb{P}(\theta_L)] = [1, 2.1, 0.8]$ for buyers, $[\kappa_L, \kappa_m, q_L^*, q_H^*, \mathbb{P}(a), \alpha_s] = [0.5, 2, 1, 2, 0.5, 0.95]$ for sellers, and $\rho = 0.5$ for the level of competition.

5 Conclusion

A prediction race has started, where firms' performance is intrinsically linked to their ability to source and utilize data, and an emerging literature (Brynjolfsson, Rock and Syverson (2019), Cockburn et al. (2019), Trajtenberg (2019), Goldfarb (2019)) argues that many of the technologies leveraged in this race are indeed general-purpose technologies (GPT) i.e. "widely used, capable of ongoing technical improvement, and enabling innovation in application sectors" (Bresnahan et al. (1995), Bresnahan (2010)). Since the consequences of this shift are likely to be deep, pervasive, and persistent, research in this area is fundamental and urgent.

Motivated by the size of the retail sector and its intense reliance on data for product and pricing decisions, I answer a "crucial" question posed by Bergemann and Bonatti (2019) in their review of markets for information: "what are the implications of acquiring an advantage in a downstream market by means of better data (e.g. improvements in the predictive power of an algorithm)?" The problem is addressed in a model that features sellers with predictive skill that has strategic value, as they face adverse selection from buyers and imperfect competition from peers. Aligning with empirical documentation of broad endemic differences in firms' use of predictive technologies, as well as forward-looking policy concerns about these disparities, I allow the precision of firms' information to be heterogeneous. I find that precision is generically efficient but redistributive. On the demand side, it tends to benefit (hurt) low- (high-) valuation buyers. On the supply side, unsurprisingly, precision allows firms to obtain greater profits, but, more interestingly, it can even benefit competitors. This highlights the importance of nuanced analysis, standing in contrast to the popular narrative of consumer harm and anti-competitive effects.

Several modeling compromises leave open avenues for future research. For one, predictive skill is exogenous in the model despite indications that very interesting forces shape its initial acquisition. Second, once an initial level of predictive skill has been acquired, this reduced-form model of precision cannot speak to the important feedback loop between data and the strategic aspects that it endows sellers with. These choices constrain the model to be static in its most natural interpretation and inhibit the study of essential dynamics, such as learning within and across each side of the market, the evolution of market structure, and trends of prices as well as production. A formal characterization of these would be very helpful for understanding the factors that drive documented heterogeneity, in the short and (possibly) long run. Lastly, the naivete of buyers with respect to sellers' information could also benefit from relaxation, given its importance to outcomes in the model, as well as its role in a host of results emerging from the literature on consumer privacy. Like these agents, I use a model to understand a complicated problem; however, structural concerns have led me to sacrifice additional complexity for the sake of intelligibility. My hope is that this work contributes to an ongoing discussion that advances the precision of our understanding.

Appendix A Exogenous Quality Setting: Equilibrium Structure

I will give some additional comments on the properties of the distributions of offers, $\{F(x|\theta\}_{\theta\in\{\theta_L,\theta_H\}}$, which complement the exposition in the main text. Then, I will derive sellers' strategies in an equilibrium where offers are ordered according by the following principle: the highest prices are offered by high signal sellers, below which are any prices offered by both types of sellers (when high signal sellers must offer some pooling prices), and below which there is a region of prices offered only by low signal sellers.

A.1 Distribution of Offers

By (3.4), equilibrium distributions of offers in a low or high-valuation match - $F(x|\theta_L)$ and $F(x|\theta_H)$, respectively - are averages of each type of seller's strategy, weighted by the mass of sellers with the respective type in matches with buyers of the respective valuation. Since buyers of either valuation have a strictly positive probability of matching with any type of seller, these distributions share identical supports - their differences lie in how mass is allocated within these.

Further, so long as some buyers obtain one offer but others obtain two, the equilibrium distributions - both aggregate price distributions and, by extension, seller strategies - are atomless. Sellers always have a strictly positive probability of competing against a peer of identical type, so a price greater than its unit cost (0) cannot be an atom, because deviating with an infinitesimal discounts would sacrifice negligible profits per sale in exchange for a discrete increase in sales. Additionally, any price offer equal to the unit cost is strictly dominated by some more expensive one because the latter would generate some profits in each sale and at least generate sales in monopoly matches, ruling out an atom at the unit cost and implying that equilibrium price distributions are continuous.

Since a loss of sales is the only deterrent to a price increase, and sales are only lost by becoming more expensive than another seller's offer or a buyer's valuation, offer distributions must be locally increasing at any price in their support that is not equal to some buyer's valuation.

Proposition A.1 (Strictly Increasing Distributions)**.** *Given prices $x, x' \in \operatorname{supp}(F(x|\theta))$ such that $x < x' < \theta_L$ or $\theta_L < x < x' < \theta_H$, $F(x|\theta)$ increases strictly in the interval $[x, x']$ for $\theta \in \{\theta_L, \theta_H\}$.*

An immediate implication is that the prices offered to any buyer are distributed in at most two disjoint intervals: one formed by prices weakly below the low-valuation, $x \leq \theta_L$, and another formed by prices strictly above it, $\theta_L < x$. As such, I only need to find the endpoints of these intervals to fully characterize the support of prices.

I start with the suprema of low-trade-permitting and overall prices. Equilibrium price distributions are atomless, so sellers who offer the highest overall price only expect to obtain sales with it in monopoly matches and bid accordingly, as per Equation (3.5). Comparing the choices of the two types of sellers, the one most willing to offer a high price is the type who expects the greatest sales from it: sellers who observe a high signal. The highest equilibrium price is that chosen by a monopolist who observes a high signal. Continuing with the most expensive offer that allows trade with both types of buyers, a seller who offers the highest low-trade-permitting price only expects to sell to a low-valuation buyer without better offers from competitors, so it chooses the highest price that these buyers would accept, mainly $x = \theta_L$.

The equation that determines the unique aggregate distribution of prices $x > \theta_L$ is that of profit equality in high-valuation matches. If any such prices are offered, I have argued that θ_H is the highest one, which only allows sellers to trade in monopoly matches, so other offers that only allow trade with high-valuation buyers should be as profitable as this one,

$$\rho\theta_H = (\rho + (1-\rho)(1 - F(x|\theta_H)))\, x \tag{A.1}$$

where I implicitly apply the convexity of the separating support. An offer of θ_H maximizes (minimizes) high- (low-) valuation match profitability,

$$\begin{aligned}\Pi(x = \theta_H|\theta = \theta_L) = 0\\ \Pi(x = \theta_H|\theta = \theta_H) = \theta_H\end{aligned}$$

and, separating prices preserve these conditional profits, so they are offered by sellers who place sufficient posteriro weight on high-valuation matches.

Switching to sellers who offer pooling prices. Consider the type of sellers who offer the highest equilibrium pooling price $x = \theta_L$. If high signal sellers only offer separating prices, then it is without loss to assume that prices are strictly ordered by the seller's type: (a) the distribution of separating prices is unique Proposition 3.5, so total welfare and its distribution are invariant in shifts of mass among seller strategies that preserve it, and (b) only low signal sellers offer pooling prices. However, when high signal sellers offer pooling prices, low signal sellers' support must overlap with these. To understand why, consider the highest pooling price offered by low signal sellers, $\overline{x}^l$, and suppose that it was strictly below θ_L. There are more high signal sellers in high-valuation matches than in low-valuation matches, so if they were the only ones who bid in $[\overline{x}^l, \theta_l]$, then an offer of $\overline{x}^l$ would entail an identical drop in profits from monopoly matches with any buyer, but a strictly larger profit increase in competitive matches where the buyer's valuation was high,

$$\begin{aligned}\Pi(\theta_L|\theta_L) - \Pi(\overline{x}^l|\theta_L) = &-(\rho + (1-\rho)(1-\alpha)(1 - F^h(\theta_l))(\theta_L - \overline{x}^l)\\ &+ (1-\rho)(1-\alpha)(F^h(\theta_L)\theta_L - F^h(\theta_L)\overline{x}^l)\\ < \Pi(\theta_L|\theta_H) - \Pi(\overline{x}^l|\theta_H) = &-(\rho + (1-\rho)\alpha(1 - F^h(\theta_l))(\theta_L - \overline{x}^l)\\ &+ (1-\rho)\alpha(F^h(\theta_L)\theta_L - F^h(\theta_L)\overline{x}^l)\end{aligned} \tag{A.2}$$

Indifference of high signal sellers, which requires $\overline{x}^l$ to be less profitable in some type of match, could only be maintained then if $\overline{x}^l$ was less profitable in low-valuation matches; meaning that low signal sellers would strictly prefer to bid above $\overline{x}^l$ - a contradiction. Extending this reasoning, I rule out any interval of pooling prices that are exclusively offered by high signal sellers. Inversely, since there are more low signal sellers in low-valuation matches, high signal sellers cannot make offers below intervals of pooling offers exclusively offered by low signal buyers[20].

Proposition A.2 (Support Overlap)**.** *If only low signal sellers offer pooling prices, it is without loss to consider the unique ordered equilibrium where strategies have disjoint support* $\mathrm{supp}(F^l) < \mathrm{supp}(F^h)$. *Whereas, if high signal sellers offer pooling prices, then their lowest pooling price offer* $\underline{x}^h$ *is such that* $\mathrm{supp}(F^l) \cap \mathrm{supp}(F^h) \cap [0, \theta_L] = [\underline{x}^h, \theta_L]$.

The only way two sellers with different weights on low- and high-valuation match profitability can be indifferent over the same set of pooling prices, however, is if these maintain profits conditionally on

[20]Same monopoly match revenue loss, strictly higher compensating profit gain in competitive matches with low-valuation buyers, so the price at the top of the interval would be more profitable in high-valuation matches (and less in low-valuation matches), and be strictly preferred by high signal sellers than the bottom one.

the buyer's valuation,

$$\Pi(\theta_L|\theta) = \Pi(x|\theta) \quad \forall x \in [\underline{x}^h, \theta_L] \text{ and } \theta \in \{\theta_L, \theta_H\} \tag{A.3}$$

Note that since the aggregate distributions of separating prices $F(x|\theta)$ for $x > \theta_L$ are unique, this pair of equations also uniquely pins down the aggregate distributions of prices in the overlapping support, as $\Pi(x|\theta) = (\rho + (1-\rho)(1-F(x|\theta)))x$.

Low and high signal sellers expect different profits from an offer of θ_L, however, so the support of low signal sellers' strategy must extend below that of high signal sellers' when the supports of their strategies overlap in the pooling region,

$$\Pi(\overline{x}^h; p_L^h) = \Pi(\theta_L; p_L^h) > \Pi(\overline{x}^h; p_L^h) = \Pi(\theta_L; p_L^l)$$

Proposition A.3 (Lowest Pooling Offers)**.** *If some separating and pooling offers are made in equilibrium, the lowest are made by low signal sellers* $\text{supp}(F^l) \cap \text{supp}(F^h)^c \cap [0, \theta_L] = [\underline{x}^l, \underline{x}^h]$.

The fact that the aggregate distribution of offers $F(x|\theta)$ is uniquely determined over separating and overlapping pooling prices, then also implies that these are uniquely determined over pooling prices that only low signal sellers offer,

$$(\rho + (1-\rho)(1-F(\underline{x}^h|\theta)))\underline{x}^h = (\rho + (1-\rho)(1-F(x|\theta)))x \quad x \in [\underline{x}^l, \underline{x}^h] \text{ and } \theta \in \{\theta_L, \theta_H\} \tag{A.4}$$

A.2 Seller Strategies

Starting with the strategy of seller types that offer the (weakly) highest prices (sellers who observe high signals), the supremum of their distribution's support is as specified by Proposition 3.3. If their highest price is θ_L, I focus on the type-invariant equilibrium where all sellers have the same mixture. When every type of seller has the same strategy, however, the distribution of competitor offers in both low and high-valuation matches is precisely equal to it, so a single equation pins down this distribution $F(x)$ at every price offer; mainly, that which guarantees that every offer yields the same profits as the highest one,

$$\rho\theta = (\rho + (1-\rho)(1-F(x)))x \tag{A.5}$$

Proposition A.4 (All Pooling Type-Invariant Equilibrium)**.** *If the highest equilibrium offer is pooling, the distribution of offers in low- and high-valuation matches is,*

$$F(x) = 1 - \frac{\rho}{1-\rho}(\theta - x) \tag{A.6}$$

and it is also the strategy of both seller types.

However, when the highest equilibrium price is separating, I construct an equilibrium by allocating mass from the top down prioritizing sellers of the lowest type .ie first adding mass to the strategy of high signal sellers and only adding mass to the strategy of low signal sellers if needed.

Disjoint Support Equilibrium - I will first derive strategies for the case where high signal sellers do not pool, so the support of their strategy with that of low signal sellers is disjoint. Since seller strategies can then be ordered strictly, I derive them sequentially, starting with that of sellers who observe high signals. The lower bound on high signal sellers' offers is,

$$\rho\theta_H = (\rho + (1-\rho)\alpha)\,\underline{x}^h \implies \underline{x}^h = \frac{\rho}{\rho + (1-\rho)\alpha}\theta_H \tag{A.7}$$

where α is the mass of high signal sellers that a seller expects to face in competitive matches if the buyer is of high-valuation - it would always lose against sellers who observe low signals, since they offer lower pooling prices in these settings. The allocation of mass between the upper and lower bound is then uniquely determined by (A.1) and the aggregation condition $F(x|\theta_H) = \alpha F^h(x)$,

$$F^h(x) = \begin{cases} 1 & \forall\, x \in (\theta_H, \infty) \\ 1 - \frac{\rho}{(1-\rho)\alpha}\frac{\theta_H - x}{x} & \forall\, x \in [\underline{x}^h, \theta_H] \\ 0 & \forall\, x \in (-\infty, \underline{x}^h) \end{cases} \tag{A.8}$$

By induction, when low signal sellers offer any separating prices, their strategy is,

$$F^l(x) = \begin{cases} 1 & \forall\, x \in (\underline{x}^h, \infty) \\ 1 - \frac{\mathbb{P}(\text{sale at price } \underline{x}^h|\theta_H)}{(1-\rho)(1-\alpha)}\frac{\underline{x}^h - x}{x} & \forall\, x \in [\hat{x}, \underline{x}^h] \\ 1 - \frac{\mathbb{P}(\text{sale at price } \theta_L; p^l_L)}{(1-\rho)\mathbb{P}(p^l_L|l)}\frac{\theta_L - x}{x} & \forall\, x \in [\underline{x}^l, \theta_L] \\ 0 & \forall\, x \in (-\infty, \underline{x}^l) \end{cases} \tag{A.9}$$

$$\mathbb{P}(\text{sale at price } x^h|\theta_H) = \rho + (1-\rho)\alpha \tag{A.10}$$

$$\mathbb{P}(\text{sale at price } \theta_L; p^l_L) = \rho + (1-\rho)((1 - F(\hat{x}|\theta_H))p^l_H + (1 - F(\hat{x}|\theta_L))p^l_L) \tag{A.11}$$

where $\hat{x}$ is the lowest separating price offered by low signal sellers.

Whereas if low signal sellers only offer pooling prices, the equation for their strategy is instead,

$$F^l(x) = \begin{cases} 1 & \forall\, x \in (\theta_L, \infty) \\ 1 - \frac{\mathbb{P}(\text{sale at price } \theta_L; p^l_L)}{(1-\rho)\mathbb{P}(p^l_L|l)}\frac{\theta_L - x}{x} & \forall\, x \in [\underline{x}^l, \theta_L] \\ 0 & \forall\, x \in (-\infty, \underline{x}^l) \end{cases} \tag{A.12}$$

where the lowest price offered by a low signal seller is the lowest one in equilibrium, so it beats every other price, yields profits $\underline{x}^l$, and must be as profitable as the highest price offered by this seller,

$$\underline{x}^l = \mathbb{P}(\text{sale at price } \overline{x}^h; p^l_L)\overline{x}^h$$

Overlapping Support Equilibrium - In settings where some high signal seller offers are pooling, their strategy over pooling prices and the lowest separating price that they offer, $\hat{x}$, is given by the formulas that I've already derived, but in the overlapping pooling region $[\underline{x}^h, \theta_L]$ high and low signal sellers' strategies at each price are pinned down by condition that profits conditionally matching with a buyer of either valuation are preserved $\Pi(\theta_L|\theta) = \Pi(x|\theta)$ for $\theta \in \{\theta_L, \theta_H\}$,

$$\begin{aligned} &|\theta_H \text{ condition:} \quad (\rho + (1-\rho)\alpha)\theta_L = (\rho + (1-\rho)(\alpha(1 - F^h(x)) + (1-\alpha)(1 - F^l(x)))x \\ &|\theta_L \text{ condition:} \quad (\rho + (1-\rho)(1-\alpha))\theta_L = (\rho + (1-\rho)((1-\alpha)(1 - F^h(x)) + \alpha(1 - F^l(x)))x \end{aligned} \tag{A.13}$$

and then the strategy of low signal sellers over the pooling prices that only they offer (below the overlapping region) follows by the same formula from the disjoint support equilibrium for their pooling offers.

Appendix B Endogenous Quality Setting: Equilibrium Properties

In this section, I will discuss the technical aspects of the candidate equilibrium introduced in Section 4.3.2 and explain why many of its distinguishing properties hold in all equilibria. An even stronger result will follow that the ordered symmetric equilibrium is unique, which allows me to affirm the genericity of the comparative static analysis in Section 4.4.

B.1 Equilibrium Distributions and Orderedness

I begin by recalling the concept of an ordered equilibrium, which connects the level of indirect utility offered to low and high-valuation buyers in a menu. Put simply, in an ordered equilibrium, sellers who offer more indirect utility in the contract intended for a high-valuation buyer must do the same in the contract intended for a low-valuation one.

Definition B.1 (Orderedness)**.** *An equilibrium is said to be weakly-ordered if, for any two equilibrium menus* (u_L, u_H) *and* (u'_L, u'_H)*,*

$$(u_H - u'_H)(u_L - u'_L) \geq 0$$

When the inequality holds strictly in almost every[21] *comparison, I refer to the equilibrium as ordered.*

These two properties have also been referred to as rank preserving and strictly rank preserving in related work (including Lester et al. (2019)). Like them, I find that orderedness is necessary holds whenever a buyer's incentive compatibility constraint binds at one of the menus that is being compared.

Lemma B.1 (Ordered Equilibrium)**.** *Almost every equilibrium menu* (u_L, u_H) *featuring a binding incentive compatibility constraint is ordered when compared to another equilibrium menu* $({u_L}', {u_H}')$*. That is,*

$$(u_H - {u_H}')(u_L - {u_L}') > 0$$

The economic rationale underlying this complementarity in indirect utilities is familiar from Garrett et al. (2019) and intuitively explained in Section 4.3.2. Consider a seller who increases the indirect utility it offers to high-valuation buyers u_H. If their incentive constraint binds, this relaxes it and allows for an increase in the quality provided to buyers of low-valuation in the paired contract (q_L, x_L), as per $q_L = \frac{u_H - u_L}{\Delta\theta}$. The seller can then offer low-valuation buyers the same utility u_L, while obtaining strictly larger profits in each sale to them; however, when low sales are more profitable, the seller also bids more aggressively for them, and this is done by making them more appealing through a low utility increase ($\uparrow u_L$). The channel that creates this complementarity is, therefore, the connection between the efficiency of the contracts and the utility they offer to both buyers. Since this link is missing among dually efficient offers, at which incentive compatibility constraints are slack, they do not need to be ordered.

Orderedness simplifies the equilibrium structure substantially, and heterogeneity in sellers' interim beliefs does not alter the fundamental complementarity between providing additional utility to low

[21] Up to a measure zero set of menus.

and high-valuation buyers, but rather how interested sellers are in forfeiting high sale profitability for low one. Therefore, the heterogeneity of the posteriors *moderates* the joint progression of u_L and u_H, but does not change the correlation between these.

Theorem B.1 (Type Monotonicity). *Let (u_L, u_H) and (u'_L, u'_H) be two equilibrium menus sharing a common binding incentive compatibility constraint with $u_i < u_i{}'$. Then,*

1. *High sale profits per-match are decreasing in generosity, $\Pi_H(u_L, u_H) > \Pi_H(u'_L, u'_H)$, while low ones increase, $(u_L, u_H) < \Pi_L(u'_L, u^j_H{}')$.*
2. *If the menus are offered by sellers of respective types $p_L \neq p'_L$, then $p_L < p'_L$*

Consider the menu $(u_L(p), u_H(p))$ occupying the p^{th} generosity-percentile in the equilibrium distribution. Then, $u_H(p)$ increases enough for profits from high sales $\Psi_H(p)(S^*_H - u_H(p))$ to decrease, but $u_L(p)$ grows passively enough to not undo the additional profitability of profits from low sales. The relationship between low/high trade profits per-match and generosity creates an equilibrium structure where seller types comparatively more interested in profits from high sales make offers that are less generous towards buyers with either valuation than those made by seller types comparatively more interested in profits from low sales. In particular, sellers relatively more convinced that they face high-valuation buyers will aim to depress bids as much as possible so as to extract these buyers' (information) rents, whereas sellers who are relatively more convinced that they face low-valuation buyers give greater consideration to capturing profitable trade with them and cede additional rents to both low and high-valuation buyers to do so.

Efficiency gains thus far have been described as taking place within low trade, implicitly treating the incentive constraint of high-valuation buyers as the only relevant one. In fact, this is a necessary property of equilibria in any economy where sellers' costs satisfy Assumption 4.2. Furthermore, in these, offers are grouped in two sets of menus. The most rationed menus, at which high-valuation buyers' incentive constraints binds, are also the least generous, and then any that are dually efficient are also more generous towards low and high-valuation buyers.

Lemma B.2 (Stacking). *The menus offered in an equilibrium where firms' costs satisfy Assumption 4.2 are partitioned into separate incentive compatibility regions such that*

1. *low-valuation buyers' incentive constraint does not bind in at any menu.*
2. *If some dually efficient menus are offered, there exists a u^{de}_i such that all utilities $u_i < u^{de}_i$ are offered in menus at which high-valuation buyers' incentive constraint binds, whereas all utilities $u_i \geq u^{de}_i$ are offered in dually efficient menus.*

Consider the logic that drives this, from the least generous bid to the most generous. The least generous menu is offered by a seller who only expects to sell if it is in a lone match, so it offers exactly the menu it'd choose if it was a monopolist with the same assessment of the buyer's probable valuation, and a monopolist would never offer a menu at which low-valuation buyers' incentive constraint binds - featuring $u_H - u_L > q^*_H \Delta\theta$ - since they could strictly increase profits from high sales by offering fewer rents to high-valuation buyers. Competition seller types who make more generous offers drives the efficiency of these alongside their generosity (through $u_H - u_L$ growth). When there is sufficient upward pressure on generosity/efficiency, such that the utility gap reaches $u_H - u_L = q^*_L \Delta\theta$, menus become dual efficient. Without an efficiency benefit to high rent concession ($\uparrow u_H$) or efficiency loss to low rent extraction ($\downarrow u_L$) among dually efficient offers, the growth of $u_H - u_L$ slows so that low-valuation buyers' incentive constraint never binds.

The results I have covered so far do not rely on the differentiability in any way, but it is a convenient feature to convey intuition and maintain tractability. It is even better to be able to work with continuously differentiable conditional distributions $F_i(u_i|\theta_i)$ over the utilities offered by the average seller to each type of buyer. Fortunately, equilibria also have these properties.

Lemma B.3 (Equilibrium Distributions)**.** *Equilibrium distributions $F_i(u_i|\theta_i)$ for $i \in \{l, h\}$.*

1. *Do not have atoms in their supports Υ_i.*
2. *Have a convex, connected low support $\Upsilon_L = [\underline{u}_L, \overline{u}_L]$. The high support Υ_H is the union of at most two convex sets disjoint sets, composed of the high utilities offered in menus where high-valuation buyers' incentive constraint bids and is slack, respectively. Furthermore, suprema over utilities always satisfy $\overline{u}_i \leq S_i^*$.*
3. *Are continuously differentiable on the interior of their supports with one-sided derivatives at the boundaries.*

Atoms make it possible for sellers to obtain discrete increases in sales in exchange for infinitesimal discounts in profits per sale, so it is clear that these cannot exist. Given that low-valuation buyers' incentive constraint does not bind in equilibrium, gaps in the low support would allow the sellers offering a menu with implied utility at the top of the gap to increase their profitability in low-valuation matches by decreasing the rents that the menu offers to low-valuation buyers, which would achieve identical low sales but strictly higher profits in each one. The logic for the convexity claim among high offers depends on whether the point u_H is such that menus that offer it are dually efficient or constrained. In the former case, high-valuation buyers' incentive constraint is slack, so a gap below any such utility would allow a seller that offers it to become more profitable by lowering these high rents, which would preserve high sales, increase high profits per sale, and not affect the efficinecy/profitability of its low-valuation buyer sales. In the latter case, a constrained menu is ordered when compared with any other, so a gap on the high side is either accompanied by one on the low side (which I have ruled out), an atom at its low utility offer (which I have ruled out), or a situation there are two constrained menus (with the same low utility term, but one has the high utility term at the top of the gap and the other the one at the bottom) and low sales vary locally in low generosity in such a way that it is not preferrable to alter the low offer whether the menu is more or less profitable in each low sale, which is not possible. Lastly, the differentiability claims follow because sellers' indifference must be maintained by the probability of winning in combination with profits per sale, and additively separable utilities allowed me to rewrite profits per sale as $S_i(u_L, u_H) - u_i$, which is smooth in marginal changes to either utility, so differentiability of equilibrium distributions becomes necessary to rule out infinitesimal deviations.

B.2 Equilibrium System of Equations Derivation

I now briefly derive the system of equations that allow me to obtain the candidate equilibrium's analytical closed form. Recall that the problem of a type $p_L^{e,j}$ seller is to offer a menu (u_L, u_H) that maximizes her expected profits,

$$\Pi^{e,j}(u_L, u_H) = \sum_{i=l,h} p_i^{e,j} \Psi_i(u_i)(S_i(u_L, u_H) - u_i)$$

subject to the constraint $u_H \geq u_L \geq 0$. The first-order conditions of this problem highlight the interdependence between the optimal amount of utility extended to each type of buyer, as well as the role of posteriors in determining the relative importance of various trade-offs. Based on the fact that only high-valuation buyers' incentive constraint can bind in the candidate equilibrium, the seller's optimality conditions are

$$\frac{\partial}{\partial u_L}: \quad \underbrace{p_L^{e,j}(1-\rho)f_L(u_L|\theta_L)(S_L(u_L,u_H)-u_L)}_{\text{sales gains}} - \underbrace{p_L^{e,j}\Psi_L(u_L)}_{\text{rent losses}} + \underbrace{p_L^{e,j}\Psi_L(u_L)\frac{\partial S_L}{\partial u_L}(u_L,u_H)}_{\text{efficiency losses}} = 0 \tag{B.1}$$

$$\frac{\partial}{\partial u_H}: \quad \underbrace{p_H^{e,j}(1-\rho)f_H(u_{h|\theta_H})(S_H^*-u_H)}_{\text{sales gains}} - \underbrace{p_H^{e,j}\Psi_H(u_H)}_{\text{rent losses}} + \underbrace{p_L^{e,j}\Psi_L(u_L)\frac{\partial S_L}{\partial u_H}(u_L,u_H)}_{\text{efficiency gains}} = 0 \tag{B.2}$$

Similar terms appear in both equations. The first two capture a typical trade-off between expected sales versus rents per-sale. By increasing indirect utility u_i, a seller makes her offer more attractive to θ_i valuation buyers, thus increasing the probability of selling to them by the mass of equilibrium menus that it would be preferred over in contested matches, mainly $p_i^{e,j}(1-\rho)f_i(u_i|\theta_i)$, whereas the cost of surrendering said rents is directly proportional to the likelihood of trading $p_i^{e,j}\Psi_i(u_i)$ with buyers of this valuation. The third term determines the efficiency effect of an increase in generosity towards θ_i valuation buyers (u_i), and it stems from the point that univariate changes in generosity u_i alter the difference in offered utilities $u_H - u_L$, which drives efficiency. When high-valuation buyers' incentive constraint binds at a menu, generosity towards low-valuation buyers (u_L increases) requires further rationing (q_L decrease), thereby reducing the gains of trade with them $S_L(u_L,u_H)$ and, by extension, the profitability of their purchases; the opposite holding for generosity towards high-valuation buyers. In other words, generosity in the low (high) offer has an efficiency cost (benefit), when high-valuation buyers' incentive constraint is locally binding. Whereas if low-valuation buyer's incentive constraint is slack, the efficiency term disappears and the only consideration for the seller is the aforementioned trade-off between the from sales and rents given to buyers of the same valuation.

The implicit objects of immediate interest are the marginal conditional utility distributions $F_i(u_i|\theta_i)$, which shape the nature of competition. Marginals have densities that measure the mass at points on the supports of some seller types' mixed strategy. These supports are atomless, convex, and monotone in the seller's type (overlapping only among any dually efficient bids). Locally, each conditional marginal density $f_i(u_i|\theta_i)$, therefore, corresponds to a weighted density of each seller type's conditional marginal density. In particular, if the utility is offer in an inefficient menu, then these is a unique seller type $p_i^{e,j}$ that offers it and the conditional marginal density is given by,

$$f_i(u_i|\theta_i) = \mu(e)P^e(j|\theta_i)f_i^{e,j}(u_i|\theta_i)$$

For utilities that are offered in dually efficient menus, the conditional marginal density still takes a weighted average form but there is a much simpler way to solve for the utility offers, so I will not use that relation.

To obtain the distribution over utilities in inefficient menus, I will further rewrite the first-order conditions of sellers offering these by applying additional equilibrium properties. In particular, recall that menus are ordered, so the particular ones of seller types $p_L^{e,j}$ who make inefficient offers are as well, and can be written as functions $(u_L(Q), u_H(Q))$ of the menu's generosity quantile Q in seller type $p_L^{e,j}$'s mixed strategy. This allows me to apply the inverse function theorem and link the conditional marginal densities $f_i(u_i|\theta_i)$ to the progression of utilities,

$$f_i(u_i|\theta_i) = \frac{\mu(e)P^e(j|\theta_i)}{\dot{u}_i^{e,j}(Q)} \tag{B.3}$$

so the conditional marginal distributions take the form,

$$F_i(u_i|\theta_i) = \mu(e)P^e(j|\theta_i)u_i^{e,j,(-1)}(u_i) + \sum_{\substack{e',j' \\ \text{s.t. } p^{e',j'} < p^{e,j}}} \mu(e')P^{e'}(j'|\theta_i) \tag{B.4}$$

where $u_i^{e,j,(-1)}(\cdot)$ is understood to be the inverse of the strictly monotone functions $u_i^{e,j}(Q)$. And, the Q^{th} quantile menu from a $p_L^{e,j}$ type obtains average sales per match,

$$\Psi_i^{e,j}(Q) = \rho + (1-\rho)F_i(u_i^{e,j}(0)|\theta_i) + (1-\rho)\mu(e)P^e(j|h)Q$$

in matches with θ_i valuation buyers.

Substituting (B.3) and (B.4) into the first-order conditions produces a standard system of ordinary differential equations that pins down the indirect utilities offered in constrained equilibrium menus. Piecewise linear costs make marginal efficiency effects locally constant, which decouples these equations and allows me to obtain analytical solutions: the equation that drives high utility $u_H^{e,j}(Q)$ is independent, under piecewise linear costs, and I can then substitute its solution into the equation driving the progression of low utility $u_L^{e,j}(Q)$. Specifically, note the marginal efficiency term becomes,

$$\frac{\partial S_L}{\partial u_H}(u_L, u_H) = (\theta_L - \kappa_L)\frac{\partial q_L}{\partial u_H} = \frac{\theta_L - \kappa_L}{\Delta\theta}$$

So, the differential system governing the progression of utilities in inefficient menus offered by a type $p_L^{e,j}$ seller is,

$$\dot{u}_L^{e,j}(Q)\left[-\frac{\theta_L - \kappa_L}{\Delta\theta} - 1\right]\Psi_L^{e,j}(Q) + (1-\rho)\mu(e)P^e(j|l)(S_L(u_L^{e,j}(Q), u_H^{e,j}(Q)) - u_L^{e,j}(Q)) = 0 \tag{B.5}$$

$$\dot{u}_H^{e,j}(Q)\left[\frac{p_L^{e,j}}{p_H^{e,j}}\frac{\theta_L - \kappa_L}{\Delta\theta}\Psi_L^{e,j}(Q) - \Psi_H^{e,j}(Q)\right] + (1-\rho)\mu(e)P^e(j|h)(S_H^* - u_H^{e,j}(Q)) = 0 \tag{B.6}$$

If offers become dually efficient, either among the offer seller type or because this seller type prefers to jump right to making dually efficient offers when it transitions from those of the adjacent seller type below, I will solve for the remaining utility offers with a different equation. As for the exact form of the utilities that solve the system (B.5)-(B.6), I define $\Xi^{e,j} = \frac{p_H^{e,j}\mathbb{P}(j|h)}{p_H^{e,j}\mathbb{P}(j|h) - p_L^{e,j}\mathbb{P}(j|l)\frac{\theta_L - \kappa_L}{\Delta\theta}}$ to tighten the expressions and write the high utility term as,

$$u_H^{e,j}(Q) = S_H^* - C_H^{e,j}\left(p_H^{e,j}\Psi_H^{e,j}(Q) - p_L^{e,j}\Psi_L^{e,j}(Q)\frac{\theta_L - \kappa_L}{\Delta\theta}\right)^{-\Xi^{e,j}} \tag{B.7}$$

with,

$$C_H^{e,j} = \frac{S_H^* - u_H^{e,j}(0)}{\left(p_H^{e,j}\Psi_H^{e,j}(0) - p_L^{e,j}\Psi_L^{e,j}(0)\frac{\theta_L - \kappa_L}{\Delta\theta}\right)^{-\Xi^{e,j}}} \tag{B.8}$$

which simplifies to,

$$u_H^{e,j}(Q) = S_H^* - (S_H^* - u_H^{e,j}(0))\left(\frac{p_H^{e,j}\Psi_H^{e,j}(0) - p_L^{e,j}\Psi_L^{e,j}(0)\frac{\theta_L - \kappa_L}{\Delta\theta}}{p_H^{e,j}\Psi_H^{e,j}(Q) - p_L^{e,j}\Psi_L^{e,j}(Q)\frac{\theta_L - \kappa_L}{\Delta\theta}}\right)^{\Xi^{e,j}} \tag{B.9}$$

Substituting this explicit form of high utility into the differential equation for low utility, I then obtain its functional form,

$$u_L^{e,j}(Q) = \frac{\Psi_L^{e,j}(0)}{\Psi_L^{e,j}(Q)} \left(C_L^{e,j} + \frac{(1-\rho)\mu(e)P^e(j|l)}{\Psi_L^{e,j}(0)} \frac{\theta_L - \kappa_L}{\theta_H - \kappa_L} \left(\frac{C_H^{e,j}}{-p_H^{e,j} P^e(j|h)} \right. \right. \tag{B.10}$$

$$* \left(\left(p_H^{e,j} \Psi_H^{e,j}(Q) - p_L^{e,j} \Psi_L^{e,j}(Q) \frac{\theta_L - \kappa_L}{\Delta\theta} \right)^{1-\Xi^{e,j}} - \left(p_H^{e,j} \Psi_H^{e,j}(0) - p_L^{e,j} \Psi_L^{e,j}(0) \frac{\theta_L - \kappa_L}{\Delta\theta} \right)^{1-\Xi^{e,j}} \right) \left. \left. \right) \right) \tag{B.11}$$

with $C_L^{e,j} = u_L^{e,j}(0)$. This heavy expression is not very informative, but I can derive an implicit form of the low utility term - as a function of both its generosity quantile Q and the high utility it is paired with $u_H^{e,j}(\cdot)$ - that is quite helpful. To do this, I simplify (B.6) and rewrite it as,

$$\Psi_L^{e,j}(Q)\dot{u}_L^{e,j}(Q) + \psi_L^{e,j}(Q)u_L^{e,j}(Q) = \psi_L^{e,j}(Q)\frac{\theta_L - \kappa_L}{\theta_H - \kappa_L}u_H^{e,j}(Q)$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}Q}\left(\Psi_L^{e,j}(Q)u_L^{e,j}(Q)\right) = \psi_L^{e,j}(Q)\frac{\theta_L - \kappa_L}{\theta_H - \kappa_L}u_H^{e,j}(Q)$$

so that,

$$u_L^{e,j}(Q) = u_L^{e,j}(0)\frac{\Psi_L^{e,j}(0)}{\Psi_L^{e,j}(Q)} + \frac{\theta_L - \kappa_L}{\theta_H - \kappa_L}\int_0^Q \frac{\psi_L^{e,j}(x)}{\Psi_L^{e,j}(Q)}u_H^{e,j}(x)\,\mathrm{d}x \tag{B.12}$$

This expression highlights the fact that the low offer is a conditional expectation of the high offers made in less generous menus. This characterization provides an immediate proof for the point that the difference in utilities, $u_H^{e,j}(Q) - u_L^{e,j}(Q)$, and hence efficiency, increases with respect to generosity, and allows me to more easily think about the response of buyer welfare to parameter perturbations, by focusing on the response of high-valuation buyer surplus and then averaging these to get that of low-valuation buyers.

The initial condition of adjacent types $p_L^{e,j} < p_L^{e',j'}$ depends on whether (a) high-valuation buyers' incentive constraint binds at most generous menu $(u_L^{e,j}(1), u_H^{e,j}(1))$ of the lower type $p_L^{e,j}$ and (b) when that happens, whether the next type of seller would prefer for their least generous bid to also inefficient or dually efficient. This decision is determined by the efficiency gain from additional high rents on the profitability of the least generous low offer,

$$\begin{aligned} &- p_H^{e',j'}\Psi_H^{e',j'}(0) + p_L^{e',j'}\Psi_L^{e',j'}(0)(\theta_L - \kappa_L)\frac{\partial q_L}{\partial u_H} \\ &= -p_H^{e',j'}\Psi_H^{e',j'}(0) + p_L^{e',j'}\Psi_L^{e',j'}(0)\frac{\theta_L - \kappa_L}{\Delta\theta} \end{aligned}$$

When this is > 0, sellers of the higher type $p_L^{e',j'}$ prefer for their lowest bid to be dually efficient, and there is a discontinuous jump in generosity between $u_H^{e,j}(1) < u_H^{e',j'}(0)$. When the condition is < 0, instead, sellers of the higher type prefer for their least generous offers to also be inefficient, and there is no discountinuity between $u_H^{e,j}(1) < u_H^{e',j'}(0)$. In either case, the progression of the low utility terms is continuous and the higher seller type's least generous low utility offer is exactly the most generous one of the lower seller type, $u_L^{e,j}(1) = u_L^{e',j'}(0)$.

I will prove that the difference in utilities $u_H^{e,j}(Q) - u_L^{e,j}(Q)$ increases in the seller's quantile and hence the efficiency of the offer, so if these reach the point $u_H^{e,j}(Q) - u_L^{e,j}(Q) = q_L^*\Delta\theta$ such that the menu becomes dually efficient, then I solve for the dually efficient menus with a simpler equation. In particular, given the utility pair $(u_L^{de}(0), u_H^{de}(0))$ at which menus become dually efficient and the mass of sellers that makes constrained offers in θ_i matches $F_i(u_i^{de})$, then the equations that low and high offers must satisfy for dually efficient bids to have the necessary property of being equally profitable in low (high) matches is,

$$(\rho+(1-\rho)F_i(u_i^{de}(0)|\theta_i))(S_i^*-u_i^{de}(0))) = (\rho+(1-\rho)(1-F_i(u_i^{de}(0)|\theta_i))Q)(S_i^*-\underline{u}_i^{de}(Q)) \quad \text{for } i\in\{l,h\} \quad \text{(B.13)}$$

where Q is the quantile among dually efficient menus of the utility $u^{de}(Q)$, and $1 - F_i(u_i^{de}(0)|\theta_i)$ is the probability that a seller makes a dually efficient offer in a θ_i match. By construction, these indirect utility functions have all the properties stipulated earlier: strictly increasing in generosity, monotone in seller type, identically ranked by their low and high utility offerings, and jointly forming a bottom pair of intervals comprised of utilities offered in menus where high-valuation buyers' incentive constraint binds, potentially followed above by another pair of intervals comprised of utilities offered in dually efficient menus.

Appendix C Exogenous Quality: Additional Proofs

Proof of Proposition 3.11. By (3.7), some high signal sellers extend separating offers if Assumption 3.3 holds and,

$$p_H^h\rho\theta_H \leq \Big(\rho+(1-\rho)(p_H^h\alpha+p_L^h(1-\alpha))\Big)\theta_L$$

Note that the left-hand term (high signal sellers' expected profits from separating offers) and the right-hand term (high signal sellers' expected profits from a pooling offer that beats all high signal sellers) are both linear in ρ, and that the former is below (above) the latter at $\rho = 0$ ($\rho = 1$), so there exists a unique level of competition $\hat{\rho}$ such that for all $\rho < \hat{\rho}$, low signal sellers strictly prefer pooling offers. I pin down this competition threshold by setting the inequality as an equality and solve for it as,

$$\hat{\rho}(\alpha,\vec{\theta},\mathbb{P}(\theta_L)) = \frac{(p_H^h-p_L^h)\alpha+p_L^h}{(p_H^h-p_L^h)\alpha+p_H^h\left(\frac{\theta_H}{\theta_L}-1\right)}$$

which decreases in $\frac{\theta_H}{\theta_L}$ and, since $\theta_L < p_H^h\theta_H$ by Assumption 3.3, also in $\mathbb{P}(\theta_H)$, which increases (decreases) p_H^h (p_L^h).

Next, I leverage the fact that only high signal sellers separate in these economies to derive the expression for the probability that a low-valuation buyer cannot trade, $1-F(\theta_L|\theta_L)$. Naturally, if high signal sellers only extend separating offers To do this, I use three equations. The first equation is from the condition that a high signal seller be indifferent between a separating prices (in particular θ_H) and the highest pooling price,

$$p_H^h\rho\theta_H = (\rho+(1-\rho)(p_H^h(1-F(\theta_L|\theta_H))+p_L^h(1-F(\theta_L|\theta_L))))\theta_L$$

The second is from the condition that only high signal sellers separate in these economies,

$$(1-F(\theta_L|\theta_H))\frac{1}{\mathbb{P}(h|\theta_H)} = (1-F(\theta_L|\theta_L))\frac{1}{\mathbb{P}(h|\theta_L)} \implies (1-F(\theta_L|\theta_H)) = \frac{\alpha}{1-\alpha}(1-F(\theta_L|\theta_L))$$

where the probability that a seller separates in a θ match is the probability of matching with a high signal seller, $\mathbb{P}(h|\theta)$, times the probability that it makes a separating offer, $1-F^h(\theta_L)$. The last equation is from Bayes' rule,

$$\frac{p_H^h}{p_L^h} = \frac{\alpha\mathbb{P}(\theta_H)}{(1-\alpha)\mathbb{P}(\theta_L)}$$

□

Appendix D Endogenous Quality: Equilibrium Property Proofs

Proof of Theorem 4.1. Per sale profits from θ_i valuation buyer intended contracts take the form:

$$\pi(q_i, x_i) = x_i - \phi(q_i) = (\theta_i q_i - \phi(q_i)) + (x_i - \theta_i q_i) = S_i(q_i) - u_i \tag{D.1}$$

and $S_i(q_i)$ is concave, reaching a maximum at q_i^*. Due to concavity, any incentive-compatible menu featuring a low quality $q_L > q_L^*$ is strictly dominated by one with a revised low contract of $(q_L^*, u_L - \theta_L(q_L - q_L^*))$, while any incentive-compatible menu featuring a high quality $q_H < q_H^*$ is strictly dominated by one with a revised high contract of $(q_H^*, u_H + \theta_H(q_H^* - q_H))$. In other words, the optimal incentive-compatible menus feature $q_L \leq q_L^*$ and $q_H \geq q_H^*$.

Furthermore, IC_i must bind[22] if the $\theta_{\neg i}$ intended contract is not efficient; else, the revised contract $(\tilde{q}_{\neg i}, \tilde{x}_{\neg i})$ featuring[23] $\tilde{q}_{\neg i} = \frac{u_H - u_L}{\Delta\theta}$ and $\tilde{x}_{\neg i} = \theta_{\neg i}\tilde{q}_{\neg i} - u_{\neg i}$ could be paired with the former θ_i contract for a strictly dominant menu - same expected $\theta_{\neg i}$ sales (by preserving the utility that a $\theta_{\neg i}$ valuation buyer obtains), strictly higher profits-per-sale in $\theta_{\neg i}$ matches (by preserving $u_{\neg i}$ and increasing the gains from trade (see (D.1))), and maintaining the incentive compatibility of θ_i buyers ($\theta_i \tilde{q}_{\neg i} - \tilde{x}_{\neg i} = u_i$). Inversely, since only one constraint IC can bind in a given menu, profit maximality implies that any menu featuring an inefficient $q_{\neg i}$ offer, also features efficient $q_i = q_i^*$ quality provision.

Incentive compatibility bounds the qualities offered in each contract by the suggested ratio: $q_L \leq \frac{u_H - u_L}{\Delta\theta}$ (to satisfy IC_H) and $q_H \geq \frac{u_H - u_L}{\Delta\theta}$ (to satisfy IC_L). I have shown that IC_i binds when $q_{\neg i} \neq q_i^*$ though, so equality of the respective bound yields the form of the inefficient quality $q_{\neg i} = \frac{u_H - u_L}{\Delta\theta}$. Lastly, any menu featuring efficient quality provision in both contracts must feature utility offerings satisfying $q_L^* \Delta\theta \leq u_H - u_L \leq q_H^* \Delta\theta$.

I conclude that: (1) menus featuring inefficient low-valuation buyer provision are IC_H binding, feature $q_L = \frac{u_H - u_L}{\Delta\theta} < q_L^*$, and are paired with efficient high-valuation buyer contracts, (2) menus featuring inefficient high-valuation buyer provision are IC_L binding, feature $\frac{u_H - u_L}{\Delta\theta} = q_H < q_H^*$, and are paired with efficient low-valuation buyer contracts, and (3) doubly efficient menus correspond to those for which $q_L^* \Delta\theta \leq u_H - u_L \leq q_H^* \Delta\theta$ almost all of which have locally slack IC constraints with the exception of boundary ones satisfying $q_i^* \Delta\theta = \frac{u_H - u_L}{\Delta\theta}$ at which $IC_{\neg i}$ binds. □

Proof of Proposition 4.2. Recall that seller types with degenerate beliefs have measure zero, so the following arguments apply to almost every bid offered in a match, in particular those that would be offered by types with nondegenerate conditional beliefs.

Consider the essential infimum and supremum, $\underline{u}_i$ and $\overline{u}_i$, respectively, on the equilibrium indirect utility offered to θ_i valuation buyers. By Theorem 4.1, equilibrium menus are separating and these must correspond to the bounds on utilities extended in θ_i intended contracts, with profits $S_i(u_L, u_H) - u_i$. If $\overline{u}_i > S_i^*$, then a discrete mass of menus would feature contracts with a non-zero probability of being accepted and entail negative profits per sale. As such, $\overline{u}_i \leq S_i^*$ and I will argue that the lower bound $\underline{u}_i$ is exactly S_i^*.

Suppose that $\underline{u}_i < S_i^*$. By Theorem 4.1, optimal contracts with $u_i < S_i^*$ entail strictly positive[24] profits per sale if $q_i > 0$ and zero profits per sale if $q_i = 0$. Further, there are only two possibilities in a neighborhood of $\underline{u}_i$: either $\underline{u}_i$ is an atom, or half-open sets $[\underline{u}_i, \underline{u}_i + \delta)$ are assigned arbitrarily small mass, as $\delta \searrow 0$, by the equilibrium distribution of indirect utility offerings by competitors in θ_i

[22] It is elementary to check that only one IC constraint can bind in a menu.

[23] Where $\tilde{q}_{\neg i} < q_{\neg i}$ if $\neg i = l$ and $>$ if $\neg i = h$.

[24] If $q_i = q_i^*$, the claim follows. If $q_i \neq q_i^*$ and $u_i = S_i(q_i)$, then lowering u_i by a small $\varepsilon < u_i - \underline{u}_i$ would allow the seller to still win matches with some probability u_i and make strictly positive profits in these, as the revision would both increase the efficiency and lower the generosity of these sales.

matches $F_i(u_i|\theta_i)$.

Recall that every contract makes nonnegative profits per sale, since not trading (offering zero quality) is always an option. As such, if there were an atom at $\underline{u}_i$, the menu implied by pairing a slightly more generous $\theta_{\neg j}$ offering $u_{\neg j} + 2\varepsilon$ and $\underline{u}_i + \varepsilon$ (for $\varepsilon > 0$ small) would be strictly dominant, entailing (at worst) an arbitrarily small decrease in $\theta_{\neg j}$ per sale profits, a discrete increase in θ_i expected sales, and thus a discrete increase in θ_i profits per match. Whereas if there were no atom at the lower bound, contracts offering u_i arbitrarily close have θ_i match profits arbitrarily close to zero (they almost surely compete against a more generous seller); so a discrete mass of these is strictly dominated by a menu of the form given in the atom case.

I conclude that almost every seller selects a pair of contracts that extend implied utilities (S_L^*, S_H^*). By Theorem 4.1, the unique optimal menu that satisfies these conditions is $((q_L^*, \phi(q_L^*)), (q_H^*, \phi(q_H^*)))$.

$\square$

No Atoms. Toward a contradiction, suppose that F_H had an atom at u_H and let (u_L, u_H) be an equilibrium menu featuring this high bid generosity.

I begin by showing that $S_H(u_L, u_H) - u_H > 0$ in any equilibrium offer (u_L, u_H). Suppose not. Then it must be that $S_L(u_L, u_H) - u_L \leq 0$, otherwise offering a pooling menu of only the low-valuation buyer's contract would strictly increase the seller's expected profits - strictly positive per-sale profits from θ_H sales and strictly positive probability of being accepted ($\rho > 0$), even by low-valuation buyers. But then expected profits $\Pi \leq 0$, which contradicts seller optimization: the seller can always offer the menu $((0,0), (q_H^*, \theta_H q_H^*))$ and obtain strictly positive expected profits. With this fact, I can rule out an atom at any u_H in $\text{supp}(\Psi_H)$.

In particular, note that at any u_H with discrete mass and for any type of seller p_L,

$$\begin{aligned}
&\lim_{\varepsilon \searrow 0} \Pi(u_L + \varepsilon, u_H + \varepsilon) - \Pi(u_L, u_H) \\
&= \lim_{\varepsilon \searrow 0} \left\{ \sum_{k=l,h} p_k \Psi_k(u_k + \varepsilon)(S_k(u_L + \varepsilon, u_H + \varepsilon) - u_k - \varepsilon) - \sum_{k=l,h} p_k \Psi_k(u_k)(S_k(u_L, u_H) - u_k) \right\} \\
&= \lim_{\varepsilon \searrow 0} \big((1 - \rho)(F_H(u_H + \varepsilon) - F_H(u_H))\big)(S_H(u_L, u_H) - u_H) \\
&> 0
\end{aligned} \tag{D.2}$$

and so (u_L, u_H) would be strictly dominated by $(u_L + \varepsilon, u_H + \varepsilon)$, for some $\varepsilon > 0$.

Furthermore, $S_L(u_L, u_H) - u_L \geq 0$ in equilibrium; otherwise, this menu would be strictly dominated by one with the paired offers $(q_L, x_L) = (0, 0)$ and $(q_H, x_H) = (q_H^*, \theta_H q_H^* - u_H)$, which maintain expected high-valuation match profits and strictly those of low ones (these buyers either select a no-loss contract, $(0, 0)$, or one with strictly positive profits per sale, (q_H, x_H)).

I close by ruling out atoms among low bids. Suppose that $\text{supp}(\Psi_L)$ had an atom at u_L and let (u_L, u_H) be an equilibrium menu featuring this low bid generosity. Inequalities (D.2) rule out $S_L(u_L, u_H) - u_L > 0$, so the only possible menu with such a low offering must be one that makes zero profits in low-valuation matches $S_L(u_L, u_H) - u_L = 0$. Suppose that $u_L > 0$. If IC_L is slack, then menu with slightly less low generosity $(u_L - \varepsilon, u_H)$ is strictly dominant (low efficiency nondecreasing hence positive profits per low sale, some low sales since $\rho > 0$, high trade profitability not affected), whereas if IC_L binds, low-valuation buyers obtain utility $u_L > 0$ from the high contract, which has strictly positive profits per sale, so the seller could just pool all buyers on this contract and makes strictly positive profits in low-valuation matches as well. Lastly, if $u_L = 0$ and $S_L(u_L, u_H) - u_L = 0$, it follows (by Theorem 4.1) that $u_H = 0$, so a strictly positive mass of such (u_L, u_H) menus would give rise to an atom in $\text{supp}(\Psi_H)$. $\square$

Weak-Orderedness. Consider two equilibrium menus (u_L, u_H) and $(\tilde{u}_L, \tilde{u}_H)$, offered by sellers of (possibly equal) respective types p_L and $\tilde{p}_L$, which violate weak-orderedness; without loss, suppose that this takes place via $\tilde{u}_H > u_H$ and $u_L > \tilde{u}_L$. I proceed case-by-case, depending on the incentive compatibility constraint that binds at (u_L, u_H).

Suppose that IC_H binds at (u_L, u_H). Then,

$$\begin{aligned} \Psi_L(u_L) &> \Psi_L(\tilde{u}_L) \\ S_L(u_L, \tilde{u}_H) - S_L(u_L, u_H) &\geq S_L(\tilde{u}_L, \tilde{u}_H) - S_L(\tilde{u}_L, u_H) \geq 0 \\ S_L(u_L, \tilde{u}_H) - S_L(u_L, u_H) &> 0 \end{aligned}$$

where the second set of inequalities follows by the convexity of costs, Theorem 4.1,

$$\begin{aligned} S_L(u_L, u_H) &= \theta_L q_L(u_L, u_H) - \phi\left(q_L(u_L, u_H)\right) \\ q_L(u_L, u_H) &= \begin{cases} \frac{u_H - u_L}{\Delta\theta} & \text{if } u_H - u_L < q_L^* \Delta\theta \\ q_L^* & \text{otherwise} \end{cases} \end{aligned}$$

while the third is due to the fact that IC_H binds at (u_L, u_H) (by assumption), so the efficiency of low trade achieved by $(u_L, \tilde{u}_H)$ must be strictly greater. Thus,

$$\Psi_L(u_L)\left(S_L(u_L, \tilde{u}_H) - S_L(u_L, u_H)\right) - \Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L, \tilde{u}_H) - S_L(\tilde{u}_L, u_H)\right) > 0$$

and,

$$\Psi_L(u_L)\left(S_L(u_L, \tilde{u}_H) - u_L\right) - \Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_L\right) > \Psi_L(u_L)\left(S_L(u_L, u_H) - u_L\right) - \Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L, u_H) - \tilde{u}_L\right) \tag{D.3}$$

When facing high-valuation buyers, Theorem 4.1 indicates that,

$$\begin{aligned} S_H(u_L, u_H) &= \theta_H q_H(u_L, u_H) - \phi\left(q_H(u_L, u_H)\right) \\ q_H(u_L, u_H) &= \begin{cases} \frac{u_H - u_L}{\Delta\theta} & \text{if } u_H - u_L > q_H^* \Delta\theta \\ q_H^* & \text{otherwise} \end{cases} \end{aligned}$$

so high contracts are decreasing in efficiency with respect to the difference in utility that a menu offers to low- versus high-valuation buyers. But the weak-orderedness violation implies $\tilde{u}_H - u_L < \tilde{u}_H - \tilde{u}_L$ and $u_H - u_L < u_H - \tilde{u}_L$, so that cost convexity also implies,

$$S_H(u_L, \tilde{u}_H) - S_H(\tilde{u}_L, \tilde{u}_H) \geq S_H(u_L, u_H) - S_H(\tilde{u}_L, u_H) \geq 0$$

which is equivalent to,

$$\Psi_H(\tilde{u}_H)\left(\left(S_H(u_L, \tilde{u}_H) - \tilde{u}_H\right) - \left(S_H(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_H\right)\right) \geq \Psi_H(u_H)\left(\left(S_H(u_L, u_H) - u_H\right) - \left(S_H(\tilde{u}_L, u_H) - u_H\right)\right) \geq 0 \tag{D.4}$$

Jointly (D.3) and (D.4) yield,

$$\begin{aligned}
&p_L\left[\Psi_L(u_L)\left(S_L(u_L,\tilde{u}_H)-u_L\right)-\Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L,\tilde{u}_H)-\tilde{u}_L\right)\right]\\
&+p_H\left[\Psi_H(\tilde{u}_H)\left(S_H(u_L,\tilde{u}_H)-\tilde{u}_H\right)-\Psi_H(\tilde{u}_H)\left(S_H(\tilde{u}_L,\tilde{u}_H)-\tilde{u}_H\right)\right]\\
&>p_L\left[\Psi_L(u_L)\left(S_L(u_L,u_H)-u_L\right)-\Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L,u_H)-\tilde{u}_L\right)\right]\\
&+p_H\left[\Psi_H(u_H)\left(S_H(u_L,u_H)-u_H\right)-\Psi_H(u_H)\left(S_H(\tilde{u}_L,u_H)-u_H\right)\right]\\
&\geq 0
\end{aligned}$$

where the last inequality comes from the optimality of (u_L, u_H) for a p_L type seller. Necessarily then, at least one of the terms in brackets at the topmost expression must be > 0 and we know, by (D.4), that the second term is ≥ 0. If the first was > 0 or if it were $= 0$ (and so the second term in brackets was > 0), then any seller - including a $\tilde{p}_L$ type - would strictly prefer $(u_L, \tilde{u}_H)$ over $(\tilde{u}_L, \tilde{u}_H)$. The only alternative then is that this first bracket term is < 0 and so that the second bracket term is > 0. Given this fact, I can take advantage of (D.3), to note that the first term in brackets of the bottom expression is also < 0 and the second > 0. The latter strict inequality can only hold when IC_L binds at both $(\tilde{u}_L, u_H)$ and, since $\tilde{u}_H > u_H$, also at $(\tilde{u}_L, \tilde{u}_H)$.

However, for (u_L, u_H) to be preferred by a type p_L seller over $(u_L, \tilde{u}_H)$, which is strictly more profitable in low-valuation matches (since (u_L, u_H) is IC_H binding), the chosen menu must be strictly more profitable in the high. But then,

$$\begin{aligned}
&\Psi_H(u_H)\left(S_H(\tilde{u}_L,u_H)-u_H\right)-\Psi_H(\tilde{u}_H)\left(S_H(\tilde{u}_L,\tilde{u}_H)-\tilde{u}_H\right)\\
&\geq \Psi_H(u_H)\left(S_H(u_L,u_H)-u_H\right)-\Psi_H(\tilde{u}_H)\left(S_H(u_L,\tilde{u}_H)-\tilde{u}_H\right)\\
&>0
\end{aligned} \tag{D.5}$$

and $(\tilde{u}_L, u_H)$ is strictly better than $(\tilde{u}_L, \tilde{u}_H)$ in high-valuation matches and identical in those with buyers of low-valuation (same sales, same profits per-sale) - making it strictly preferable for a $\tilde{p}_L$ type.

These arguments are sufficient to establish that[25] for a given optimal menu (u_L, u_H) at which IC_H binds, then any other equilibrium menu $(\tilde{u}_L, \tilde{u}_H)$ with $\tilde{u}_i > u_i$ - offered by any type of seller - must have $\tilde{u}_{\neg i} \geq u_{\neg i}$.

The only comparisons left to consider are those between two IC_L binding menus and those between one at which IC_L binds one with another that is dually efficient; any comparison where IC_H binds in a menu is covered by the previous reasoning. But, if IC_L is to bind in either of the potentially non-weakly-ordered menus, giving rise to $\tilde{u}_H > u_H$ and $u_L > \tilde{u}_L$, then it must bind at the menu with the largest utility difference, $(\tilde{u}_L, \tilde{u}_H)$. Suppose that this is so and consider the alternative bid $(\tilde{u}_L, u_H)$.

By the fact that (u_L, u_H) is either dually efficient or IC_L binding, the same must be true for $(\tilde{u}_L, u_H)$ and $(u_L, \tilde{u}_H)$, which have strictly larger differences in offered utilities, so $S_L^* = S_L(u_L, u_H) = S_L(\tilde{u}_L, u_H) = S_L(\tilde{u}_L, \tilde{u}_H) = S_L(u_L, \tilde{u}_H)$ and,

$$\begin{aligned}
&\Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L,u_H)-\tilde{u}_L\right)-\Psi_L(\tilde{u}_L)\left(S_L(\tilde{u}_L,\tilde{u}_H)-\tilde{u}_L\right)\\
&=\Psi_L(u_L)\left(S_L(u_L,u_H)-u_L\right)-\Psi_L(u_L)\left(S_L(u_L,\tilde{u}_H)-u_L\right)\\
&=0
\end{aligned} \tag{D.6}$$

[25]Note that the case of IC_H binding at (u_L, u_H) and $u_H > \tilde{u}_H$ but $\tilde{u}_L > u_L$ is subsumed in one I've established, because IC_H must also bind at $(\tilde{u}_L, \tilde{u}_H)$ in these other inequalities.

Given that IC_L binds at $(\tilde{u}_L, \tilde{u}_H)$ however, $S_H(u_L, \tilde{u}_H) - S_H(\tilde{u}_L, \tilde{u}_H) > S_H(u_L, u_H) - S_H(\tilde{u}_L, u_H)$ and,

$$\begin{aligned}
&\Psi_H(u_H)\left(S_H(\tilde{u}_L, u_H) - u_H\right) - \Psi_H(\tilde{u}_H)\left(S_H(\tilde{u}_L, \tilde{u}_H) - \tilde{u}_H\right) \\
&> \Psi_H(u_H)\left(S_H(u_L, u_H) - u_H\right) - \Psi_H(\tilde{u}_H)\left(S_H(u_L, \tilde{u}_H) - \tilde{u}_H\right) \\
&\geq 0
\end{aligned} \tag{D.7}$$

where the weak inequality is due to the choice of (u_L, u_H) over $(u_L, \tilde{u}_H)$ by the p_L type seller, while the strict inequality follows from the IC_L constraint binding in all four of menus under consideration. Jointly (D.6) and (D.7) then imply that $(\tilde{u}_L, \tilde{u}_H)$ is strictly dominated by $(\tilde{u}_L, u_H)$ for any seller type however, since the latter is equally profitable in low-valuation matches and strictly better in the high. □

Ordered, Support Convexity, and Profit Ranking. I will now strengthen the claim that equilibrium menus are weakly ordered to one of (strict) orderedness. The proof is broken up into 8 steps which will show the monotonicity of profits per low- and high-valuation match with respect to generosity as well as convex supports for the probability of winning distributions Ψ_j.

Step 1 (Conditional Expected Sales): *Every type of seller expects the same sales, $\Psi_i(u_i)$ from an indirect utility offer of u_i.*

The buyer's type determines the distribution of signals observed by sellers with each precision, α_e, and so the distribution of seller types,

$$\mathbb{P}(p_L = p|\theta_i) = \sum_{e,j} \mathbb{1}\left(p = P^e(\theta_L|j)\right) P^e(j|\theta_i)\mu(e)$$

A seller's mixing distribution, $\mathbb{P}((u_L, u_H)|p_L)$, is determined by her type. As such, every seller's expected distribution of competitor bids,

$$F(\tilde{u}_i \leq u_i|\theta_i) = \sum_{p_L} \mathbb{P}(\tilde{u}_i \leq u_i|p)\mathbb{P}(p_L|\theta_i)$$

is identical when the buyer's type is θ_i.□

Step 2 (No IC_L): *Under Assumption 4.2, IC_L binding menus are not offered in equilibrium.*

The profits per high sale of any menu (u_L, u_H) are given by,

$$S_H(u_L, u_H) - u_H \leq S_H^* - u_H$$

so $u_H \leq S_H^*$, as losses are strictly increasing in u_H for larger values. The gap between utilities of any menu is therefore bounded by,

$$\begin{aligned}
u_H - u_L \leq u_H \leq S_H^* &= (\theta_H - \kappa_m)(q_H^* - q_L^*) + (\theta_H - \kappa_L)q_L^* \\
&\leq (\theta_H - \theta_L)(q_H^* - q_L^*) + (\theta_H - \theta_L)q_L^* \\
&= \Delta\theta q_H^*
\end{aligned}$$

Step 3 (Dually Efficient Characterization): *If $\Delta\theta q_L^* < \Delta u < \Delta\theta q_H^*$, then sufficiently small changes u_H or u_L leave expected profits per low- and high-valuation match unchanged. Further, the existence of a menu $(\overline{u}_L^{de}, \overline{u}_H^{de})$ with $\Delta\theta q_L^* < \overline{u}_H^{de} - \overline{u}_L^{de} \leq \Delta\theta q_H^*$ implies the existence of another menu $(\underline{u}_L^{de}, \underline{u}_H^{de})$ with $\underline{u}_H^{de} - \underline{u}_L^{de} = \Delta\theta q_L^*$ and convex regions $[\underline{u}_i^{de}, \overline{u}_i^{de}]$ for $i \in \{L, H\}$ made up of offerings from dually efficient menus that satisfy the strict inequalities. Lastly, if $\Delta\overline{u}^{de} = \Delta\theta q_H^*$, then there exists a $\delta > 0$, such that a bid of $(u_L - \delta, u_H - \delta)$ has identical profits in low- and high-valuation matches.*

Since lowering either offering does not affect the efficiency of bids when the inequalities are strict, a gap below either u_i would allow a strict increase in profitability from undercutting the original utility by some $\epsilon > 0$. Given that there aren't gaps under either coordinate then, there can't be IC_H binding menus immediately below - by weak-orderedness and these menus' strictly smaller Δu respectively - so, all coordinates $u_i' \in [u_i - \delta, u_i]$ for $\delta > 0$ small and $j \in \{l, h\}$ belong to dually efficient menus, and consequently, $\Psi_i(u_i')(S_i^* - u_i') = \Psi_i(u_i)(S_i^* - u_i)$ (otherwise, some seller would be able to deviate to a bid with higher expected profits per-match without affecting the efficiency of the paired contract). This establishes the first sentence's claim.

The preceding logic implies that dually efficient menus satisfying the pair of strict inequalities have other dually efficient menus immediately below them in generosity and that the profitability of offering these menus, conditionally on matching with either type of buyer, is identical. Consider the infimum u_i coordinates of the contiguous intervals made up by these dually efficient menus and refer to it as $\underline{u}_i^{de}$ for $i \in \{l, h\}$. Note that $\Delta\underline{u}^{de} = \Delta\theta q_L^*$, since otherwise necessarily there'd be a gap below one of the $\underline{u}_i^{de}$ coordinates (no sufficiently close dually efficient menus with coordinates below, whereas coordinates from IC_H binding menus would create gaps), which would allow an analogous deviation[26] as in the previous paragraph for those seller types offering dually-efficient menus with u_i's close to $\underline{u}_i^{de}$. This establishes the second sentence.

For the case that $\Delta\overline{u}^{de} = \Delta\theta q_H^*$, it is sufficient to observe that all entries immediately below the menu $(\overline{u}_L^{de}, \overline{u}_H^{de})$ and sufficiently close must also be dually efficient to avoid creating a gap. By continuing down distribution in either coordinate one must eventually reach a dually efficient menu with $\Delta u < \Delta\theta q_H^*$; else, there'd either be a gap (allowing the familiar deviation) if one encountered an IC_H menu before reaching a dually efficient one that satisfied the pair of strict inequalities, or it'd hold that $\underline{u}_L^{de} = 0$ with a paired high utility of $\tilde{u}_H = \underline{u}_L^{de} + \Delta\theta q_H^*$. In this last case, the lack of atoms, weak-orderedness (ruling out a non-zero mass of IC_H binding ones having entries below $\tilde{u}_H$), and lack of dually efficient bids with $\Delta u < \Delta\theta q_H^*$ would imply that there is a gap below $\tilde{u}_H$ and so that this menu's profitability could be strictly improved by lowering its high offering (more profits per high sale without decreasing the efficiency of low sales).

To prove the statement in the last sentence, consider the supremum over dually efficient bids u_i with $u_i < \overline{u}_j^{de}$ that belong to a menu with $\Delta u < \Delta\theta q_H^*$, and refer to it as u_j'. If $u_j' = \overline{u}_j^{de}$, we are done by the previous claims. If $u_j' < \overline{u}_j^{de}$, then all menus with bids in $[u_j', \overline{u}_j^{de}]$ for either $j \in \{l, h\}$ must have $\Delta u = \Delta\theta q_H^*$ and it is sufficient to establish the invariance of profits in this region, as the previous arguments take over for dually efficient menus with bids below. Note that the existence of a high offering u_H that allows

[26] Profits conditionally on buyer type are continuous with respect to generosity due to the lack of atoms in the bid distribution, while the continuity of profits per sale follows from the efficiency formulation $S_i(u_L, u_H) - u_i$.

greater expected profits per high-valuation match than neighboring ones above would allow these sellers above to strictly increase their expected profits per high-valuation match (while leaving low-valuation match ones unperturbed), by instead choosing u_H as the high pairing; by extension of this local argument, expected high-valuation match profits are maintained by menus satisfying $\Delta\theta q_H^*$ in the interval $[u_H', \overline{u}_H^{de}]$. The invariance of expected low-valuation match profits for menus with bids in $[u_L', \overline{u}_L^{de}]$ follows by a similar logic: the existence of a point u_L in this interval allowing larger expected low-valuation match profits would allow sellers above to shift their high and low offering by the same amount so as to obtain the superior expected profits per low-valuation match while preserving the same expected profits per high-valuation match.

Step 4 (Convex Ψ_L Support): *There are no gaps in* $\text{supp}(\Psi_L)$.

Since IC_L never binds in an equilibrium bid by Step 2, gaps in $\text{supp}\,\Psi_L$ would imply the existence of a deviation for sellers bidding menus featuring a u_L offer close to the top of said gap. Such a seller could decrease its low offer to some value in the gap, so as to obtain a discrete increase in profits per low sale, while sacrificing an arbitrarily small number of low sales (sales continuous in generosity) and maintaining expected profits per high-valuation match, thereby strictly increasing expected profits. The only non-obvious case is if $\Delta u = \Delta\theta q_H^*$, but then we know from Step 3 that all the u_H' immediately below u_H preserve expected profits per high-valuation match, so a revision of u_L and u_H by the same δ downward would do what is described.

Step 5 (Ordereness Violations): *The only possible violation of strict-orderedness between two equilibrium menus (u_L, u_H) and (u_L', u_H'), where IC_H binds in at least one of them, is if $u_L = u_L'$, $u_H' > u_H$, and $p_L' > p_L$ for the respective seller types that offer these.*

Given two equilibrium menus (u_L, u_H) and (u_L', u_H'), such that IC_H binds in at least one, these must be weakly ordered, so the violation of strict ordering must involve either $u_L = u_L'$ and $u_H > u_H'$ or $u_H = u_H'$ and $u_L > u_L'$. However, the second case cannot be.

To see this point, note that if $u_H = u_H'$ and $u_L > u_L'$, IC_H must bind at the menu with the smaller generosity difference, (u_L, u_H). As such, any other bid $(\tilde{u}_L, \tilde{u}_H)$ with $\tilde{u}_L \in [u_L', u_L]$ is weakly ordered with respect to this menu, so $\tilde{u}_H \leq u_H$. And if IC_H bound at $(\tilde{u}_L, \tilde{u}_H)$, then it too would be weakly ordered with respect to (u_L', u_H'), so that $u_H' \leq \tilde{u}_H$. Jointly, these statements imply that $u_H = \tilde{u}_H = u_H'$ for any IC_H binding bid with a low offering in interval $[u_L', u_L]$ and thus that a nonzero mass of such menus gives rise to an atom at u_H (contradicting the lack of atoms). I only have the mass of dually efficient menus left to fill the intervals below u_L. Weak ordering requires these to satisfy $\tilde{u}_H \leq u_H$ and dual efficiency $\Delta\theta q_L^* \leq \Delta\tilde{u}$, so that,

$$\tilde{u}_L \leq \tilde{u}_H - \Delta\theta q_L^* \leq u_H - \Delta\theta q_L^* < u_L$$

Implying the existence of a $\delta > 0$, for which $\Psi_L([u_L - \delta, u_L]) = 0$ (contradicting Step 4).

If $u_L' = u_L$ and $u_H' > u_H$ instead, IC_H must bind at (u_L, u_H); therefore, (u_L', u_H') is strictly more profitable in low-valuation matches (more efficient, same low generosity, same sales). The menu (u_L, u_H) is offered in equilibrium though, so it cannot be strictly dominated and must be superior to (u_L', u_H') in high-valuation matches. This difference in profits conditionally on a buyer type can only be optimal for two sellers if they differ in seller type, $p'_L \neq p_L$, with the one who places more weight on low-valuation matches $p'_L > p_L$ offering the menu (u'_L, u'_H) that is more profitable in them.

Step 6 (Profit Ranking): *Given two menus (u_L, u_H) and (u'_L, u'_H) with $u_L < u'_L$, it must be that the expected profits per high-valuation match satisfy $\Psi_H(u'_H)(S^*_H - u'_H) \leq \Psi_H(u_H)(S^*_H - u_H)$ and inversely for low-valuation match profits, $\Psi_L(u'_L)(S_L(u'_L, u'_H) - u'_L) \geq \Psi_L(u_L)(S_L(u_L, u_H) - u_L)$, with both inequalities either strict or equal. In particular, if the inequalities are strict, the type p'_L of the seller who bids (u'_L, u'_H) must be strictly greater than that of the seller who bids (u_L, u_H).*

Consider menus (u_L, u_H) and (u'_L, u'_H) with $u_L < u'_L$ close. If $\Delta\theta q^*_L \leq \Delta u' \leq \Delta\theta q^*_H$, any (u_L, u_H) with u_L sufficiently close must be dually efficient and maintain expected profits per high and low-valuation match by Step 3. Whereas if $\Delta u' < q^*_L \Delta\theta$, for u_L close: (a) $\Pi_H(u'_L, u'_H) = \Pi_H(u_L, u'_H)$, (b) to rule out a deviation toward the more efficient bid (u_L, u'_H) by the type bidding (u_L, u_H), necessarily $\Pi_H(u_L, u'_H) < \Pi_H(u_L, u_H)$, and (c) to rule out a deviation, by the type offering (u'_L, u'_H), towards the more high-valuation match profitable bid (u_L, u_H), necessarily $\Pi_L(u'_H, u'_L) > \Pi_L(u_H, u_L)$. The menu with the more generous low offer is therefore more profitable in low-valuation matches and must be offered by the seller of type p_L, which places more weight on low-valuation matches. This local reasoning around any point $\text{supp}(\Psi_L)$ yields the global claim.

$\square$

Continuous Differentiable Distributions. I will show that the functions $\Psi_H(\cdot)$ and $\Psi_L(\cdot)$ are continuously differentiable. Since these functions are given by the composition of F_H and F_L with a continuous mononote function, this will prove the continuous differentiability of the equilibrium offer distributions.

I present the case of Ψ_H (Ψ_L's is analogous). Let u_H be a utility level offered in the interior of the support of F_H and u_L be its paired low utility offering such that the menu (u_L, u_H) is optimal for some seller of type p_L. I proceed to bound the difference $\Psi_H(u_H + \varepsilon) - \Psi_H(u_H)$ from above and below.

For any $\varepsilon \in \mathbb{R}$, the expected profit to this type of seller from the deviation menu $(u_L, u_H + \varepsilon)$ can be decomposed as:

$$
\begin{aligned}
& p_L \Psi_L(u_L)(S_L(u_L, u_H + \varepsilon) - u_L) + p_H \Psi_H(u_H + \varepsilon)(S_H(u_L, u_H + \varepsilon) - u_H - \varepsilon) \\
& = p_L \Psi_L(u_L)(S_L(u_L, u_H) - u_L) + p_H \Psi_H(u_H)(S_H(u_L, u_H) - u_H) \\
& + p_L \Psi_L(u_L)(S_L(u_L, u_H + \varepsilon) - S_L(u_L, u_H)) + p_H \Psi_H(u_H)(S_H(u_L, u_H + \varepsilon) - \varepsilon - S_H(u_L, u_H)) \\
& + p_H(\Psi_H(u_H + \varepsilon) - \Psi_H(u_H))(S_H(u_L, u_H + \varepsilon) - u_H - \varepsilon)
\end{aligned}
$$

and by optimality, it must be that $\Pi(u_L, u_H; p_L) \geq \Pi(u_L, u_H + \varepsilon; p_L)$, which implies the following inequality,

$$
\begin{aligned}
& p_H(\Psi_H(u_H + \varepsilon) - \Psi_H(u_H))(S_H(u_L, u_H + \varepsilon) - u_H - \varepsilon) \\
& \leq p_L \Psi_L(u_L)(S_L(u_L, u_H) - S_L(u_L, u_H + \varepsilon)) + p_H \Psi_H(u_H)(S_H(u_L, u_H) - S_H(u_L, u_H + \varepsilon) + \varepsilon)
\end{aligned}
\tag{D.8}
$$

Similarly, for any $\varepsilon \in \mathbb{R}$ such that $u_H + \varepsilon$ is in the interior of the support of F_H, there exists $u_{l,\varepsilon}$ for which $(u_{l,\varepsilon}, u_H + \varepsilon)$ is the optimal bid of some seller with type $\tilde{p}_L$. And, decomposing the profits from this bid to said seller as above:

$$\begin{aligned}
&\tilde{p}_L\Psi_L(u_{l,\varepsilon})(S_L(u_{l,\varepsilon},u_H+\varepsilon)-u_{l,\varepsilon})+\tilde{p}_H\Psi_H(u_H+\varepsilon)(S_H(u_{l,\varepsilon},u_H+\varepsilon)-u_H-\varepsilon)\\
&=\tilde{p}_L\Psi_L(u_{l,\varepsilon})(S_L(u_{l,\varepsilon},u_H)-u_{l,\varepsilon})+\tilde{p}_H\Psi_H(u_H)(S_H(u_{l,\varepsilon},u_H)-u_H)\\
&+\tilde{p}_L\Psi_L(u_{l,\varepsilon})(S_L(u_{l,\varepsilon},u_H+\varepsilon)-S_L(u_{l,\varepsilon},u_H))+\tilde{p}_H\Psi_H(u_H)(S_H(u_{l,\varepsilon},u_H+\varepsilon)-\varepsilon-S_H(u_{l,\varepsilon},u_H))\\
&+\tilde{p}_H(\Psi_H(u_H+\varepsilon)-\Psi_H(u_H))(S_H(u_{l,\varepsilon},u_H+\varepsilon)-u_H-\varepsilon)
\end{aligned}$$

Again, since here $(u_{l,\varepsilon},u_H+\varepsilon)$ is optimal for a seller of type $\tilde{p}_L$, it must be that $\Pi(u_{l,\varepsilon},u_H+\varepsilon;\tilde{p}_L)\geq\Pi(u_{l,\varepsilon},u_H;\tilde{p}_L)$, which implies the following inequality,

$$\begin{aligned}
&\tilde{p}_H(\Psi_H(u_H+\varepsilon)-\Psi_H(u_H))(S_H(u_{l,\varepsilon},u_H+\varepsilon)-u_H-\varepsilon) \qquad\qquad \text{(D.9)}\\
&\geq\tilde{p}_L\Psi_L(u_{l,\varepsilon})(S_L(u_{l,\varepsilon},u_H)-S_L(u_{l,\varepsilon},u_H+\varepsilon))+\tilde{p}_H\Psi_H(u_H)(S_H(u_{l,\varepsilon},u_H)-S_H(u_{l,\varepsilon},u_H+\varepsilon)+\varepsilon)
\end{aligned}$$

So, by (D.8) and (D.9), I can form a squeezed inequality for the derivative of $\Psi_H(\cdot)$,

$$\begin{aligned}
&\frac{\frac{\tilde{p}_L}{\tilde{p}_H}\Psi_L(u_{l,\varepsilon})(S_L(u_{l,\varepsilon},u_H)-S_L(u_{l,\varepsilon},u_H+\varepsilon))+\Psi_H(u_H)(S_H(u_{l,\varepsilon},u_H)-S_H(u_{l,\varepsilon},u_H+\varepsilon)+\varepsilon)}{\varepsilon(S_H(u_{l,\varepsilon},u_H+\varepsilon)-u_H-\varepsilon))}\\
&\leq\frac{\Psi_H(u_H+\varepsilon)-\Psi_H(u_H)}{\varepsilon}\leq\\
&\frac{\frac{p_L}{p_H}\Psi_L(u_L)(S_L(u_L,u_H)-S_L(u_L,u_H+\varepsilon))+\Psi_H(u_H)(S_H(u_L,u_H)-S_H(u_L,u_H+\varepsilon)+\varepsilon)}{\varepsilon(S_H(u_L,u_H+\varepsilon)-u_H-\varepsilon)}
\end{aligned}$$

Considering the right-hand derivative, for $\varepsilon\searrow 0$ small $S_H(u_L,u_H+\varepsilon)-u_H-\varepsilon>0$, since (u_L,u_H) is an equilibrium menu and (as I've argued in Section D in the proof ruling out atoms) $S_H(u_L,u_H)-u_H>0$ for all such menus. With the nonzero denominator established, recall that $q_i(u_L,u_H)$ (specified in Theorem 4.1) is everywhere left as well as right differentiable in each variable and that $S_i(u_L,u_H)=\theta_i q_i(u_L,u_H)-\phi(q_i(u_L,u_H))$, so taking the limit of the right-hand expression, I obtain:

$$\frac{-\frac{p_L}{p_H}\Psi_L(u_L)\frac{\partial S_L}{\partial_+u_H}(u_L,u_H)+\Psi_H(u_H)\left(1-\frac{\partial S_H}{\partial_+u_H}(u_L,u_H)\right)}{S_H(u_L,u_H)-u_H}$$

The left-hand side expression of the inequality is similar to the right, with the exception that $u_{l,\varepsilon}$ is substituted in for u_L and the weighting probabilities are $(\tilde{p}_L,\tilde{p}_H)$ instead of (p_L,p_H), thus requiring a different argument. If $u_H-u_L>q_L^*\Delta\theta$, I establish in the orderedness proof that offerings locally around both u_L and u_H are those of dually efficient bids and that they preserve expected profits from both low- and high-valuation matches - $\Pi_i(u_{l,\varepsilon},u_H+\varepsilon)=\Pi_i(u_{l,\varepsilon},u_H)$ for $i\in\{l,h\}$. So, if a seller of type $\tilde{p}_L$ finds $(u_{l,\varepsilon},u_H+\varepsilon)$ optimal, then so would a seller of type p_L, yielding a limit of the left-hand side expression as above (replacing $\tilde{p}_L$ with p_L for ε small). On the other hand, if $u_H-u_L<q_L^*\Delta\theta$, IC_H binds locally for all menus with $\varepsilon>0$ small, so the strict ordering of profits by type requires $\tilde{p}_L\searrow p_L$. Lastly, if $u_H-u_L=q_L^*\Delta\theta$, then it must be possible to select a subsequence with $u_H-u_{l,\varepsilon_{\tilde{n}}}>u_H-u_L$ or $<u_H-u_L$. In the former (latter) case, IC_H is slack (binds) at the menus of the subsequence, so the argument from the $u_H-u_L>q_L^*\Delta\theta$ ($u_H-u_L<q_L^*\Delta\theta$) case applies, and this is sufficient to obtain a convergent left-hand inequality.

Therefore, the limit as $\varepsilon\to 0$ of the left-hand and right-hand inequalities is identical, thus establishing the right-hand derivative claim:

$$\frac{\mathrm{d}\Psi_H}{\mathrm{d}_+u_H}(u_H)=\frac{-\frac{p_L}{p_H}\Psi_L(u_L)\frac{\partial S_L(u_L,u_H)}{\partial_+u_H}+\Psi_H(u_H)(1-\frac{\partial S_H(u_L,u_H)}{\partial_+u_H})}{S_H(u_L,u_H)-u_H}$$

The left-hand derivative argument is analogous except that I consider the menu $(u_L, u_H - \varepsilon)$.

As stated at the start, the case of $\Psi_L(\cdot)$ is analogous, ultimately yielding,

$$\frac{\mathrm{d}\Psi_L}{\mathrm{d}u_L}(u_L) = \frac{-\frac{p_H}{p_L}\Psi_H(u_H)\frac{\partial S_H(u_L,u_H)}{\partial u_L} + \Psi_L(u_L)(1 - \frac{\partial S_L(u_L,u_H)}{\partial u_L})}{S_L(u_L, u_H) - u_L}$$

Under strictly convex costs, at the sole[27] points of possible non-differentiability where $u_H - u_L = \Delta\theta q_i^*$,

$$\frac{\partial S_i(u_L, u_H)}{\partial_+ u_i}(u_L, u_H) = \frac{\partial S_i(u_L, u_H)}{\partial_- u_i}(u_L, u_H) = \frac{\theta_i - \phi'(q_i^*)}{\Delta\theta} = 0$$

hence the stronger differentiability claim for Ψ_i. If costs instead take a piecewise form, unless $u_H - u_L = \Delta\theta q_i^*$, then $S_i(u_L, u_H)$ is locally linear (or constant) in each variable, so that the left and right derivatives are also equal. Since $S_i(u_L, u_H) - u_i$ are continuous in (u_L, u_H) and $\Psi_i(\cdot)$ are continuous by the lack of atoms, the distributions Ψ_i are also continuously differentiable on the interior of the supports with the exception of points u_i corresponding to the (no more than two) bids (u_L, u_H) at which $u_H - u_L = \Delta\theta q_i^*$. □

[27]For a given equilibrium distribution.

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